VII. *The Application of a Method of Differences to the Species of Series whose Sums are obtained by Mr.* Landen, *by the Help of impossible Quantities. By Mr.* Benjamin Gompertz. *Communicated by the Rev.* Nevil Maskelyne, *D. D. Astronomer Royal, F. R. S.*

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Having some years back, when reading the learned Mr. Landen's fifth Memoir, discovered the manner of applying a method of differences, to the species of series, whose sums are there obtained by the help of impossible quantities, and having since extended that application, I now venture to offer it to the consideration of others.

The practice of this method, in most cases, appears to me extremely simple; and on that account, I am almost induced to imagine, that they have already been considered by mathematicians; indeed since the greatest part of this Paper was written, I met with Euler's *Institutiones Calculi integralis*; two simple series are in that work summed by multiplications similar to those employed in the investigation of the principal theorems contained in this Paper; but whether that learned mathematician has farther pursued the method, in that or in any other work, I have not as yet been able to ascertain.

I have purposely considered some of the series summed by Mr. Landen, to afford an opportunity of comparing both the results and methods; and because the series may have particular cases in which both Mr. LANDEN's means and my own fail: I have added towards the end a general scholium concerning the cause, circumstances, and consequences of such failure in my method.

The foundation of the theorems depends on the following well known lemmas.

No. I.

2 sine of vz . sine of tz, is equal to
cos. of $\overline{t-v}\,.\,z$ — cos. of $\overline{t+v}\,.\,z$.

No. II.

2 sine of vz . cos. of tz, is equal to
sine of $\overline{t+v}\,.\,z$ — sine of $\overline{t-v}\,.\,z$, or
sine of $\overline{t+v}\,.\,z$ + sine of $\overline{v-t}\,.\,z$.

No. III.

2 cos. of vz . cos. of tz, is equal to
cos. of $\overline{t-v}\,.\,z$ + cos. of $\overline{t+v}\,.\,z$.

Theorem I.

If there be an infinite series a . sine of pz, + b . sine of $\overline{p+q}\,.\,z$, + c . sine of $\overline{p+2q}\,.\,z$, + d . sine of $\overline{p+3q}\,.\,z$ &c. $= s$, and from the series a, b, c, d, e, f, &c. there be continually formed new series

$$\left\{\begin{array}{l} a,\ a',\ b',\ c',\ d',\ e',\ \text{\&c.} \\ a,\ a'',\ b'',\ c'',\ d'',\ e'',\ \text{\&c.} \\ a,\ a''',\ b''',\ c''',\ d''',\ e''',\ \text{\&c.} \\ \text{\&c. \&c. \&c. \&c. \&c. \&c. \&c.} \end{array}\right\}$$

Every new series, being formed from that immediately above, by taking the differences of the terms, exactly in the same manner as in the common differential method, except that they here continually commence with the first term, a; and if p' be put $=p-\frac{1}{2}q$, $p''=p'-\frac{1}{2}q$, $p'''=p''-\frac{1}{2}q$, $p^{iv}=p'''-\frac{1}{2}q$. &c. s . 2 sine of $\frac{1}{2}qz=s'$, $-s'$. 2 sine of $\frac{1}{2}qz=s''$, s''. 2 sine of $\frac{1}{2}qz=s'''$, $-s'''$. 2 sine of $\frac{1}{2}qz=s^{iv}$ &c. Then shall

a.cos. of $p'z+a'$. cos. of $\overline{p'+q}\,.\,z+b'$. cos. of $\overline{p'+2q}\,.\,z+c'$ cos. of $\overline{p'+3q}\,.\,z+$ &c.$=s'$,
a.sine of $p''z+a''$. sine of $\overline{p''+q}\,.\,z+b''$. sine of $\overline{p''+2q}\,.\,z$ &c.$=s''$,
a.cos. of $p'''z+a'''$.cos. of $\overline{p'''+q}\,.\,z+b'''$.cos. of $\overline{p'''+2q}\,.\,z$&c.$=s'''$,
a.sine of $p^{iv}z+a^{iv}$. sine of $\overline{p^{iv}+q}\,.\,z+b^{iv}$. sine of $\overline{p^{iv}+2q}\,.\,z$&c.$=s^{iv}$,
&c, + &c,+ &c, , =, &c.

For, multiplying the series a . sine of $pz+b$. sine of $\overline{p+q}\,.\,z$ $+c$.sine of $\overline{p+2q}\,.\,z$ &c. $=s$, by 2 . sine of $\frac{1}{2}qz$ by lemma No. I. we get, a.cos. of $\overline{p-\frac{1}{2}q}\,.\,z-a$. cos. of $\overline{p+\frac{1}{2}q}\,.\,z+b$. cos. of $\overline{p+\frac{1}{2}q}\,.\,z-b$. cos. of $\overline{p+\frac{3}{2}}\,.\,qz+c$. cos. of $\overline{p+\frac{3}{2}}\,.\,qz-c$. cos. of $\overline{p+\frac{5}{2}q}\,.\,z$ &c. $=s$. 2 sine of $\frac{1}{2}qz$ ∴ putting $b-a=a'$, $c-b=b'$, $d-c=d'$, &c. $p-\frac{1}{2}q=p'$, s . 2 sine of $\frac{1}{2}qz=s'$ we have, a . cos. of $p'z+a'$. cos. of $\overline{p'+q}\,.\,z+b'$. cos. of $\overline{p'+2q}\,.\,z+c'$. cos. of $\overline{p'+3q}\,.\,z$ &c.$=s'$, multiply this by 2 sine of $\frac{1}{2}qz$ by help of lemma No. II. and we have $-a$ sine of $\overline{p'-\frac{1}{2}q}\,.\,z+a$ sine of $\overline{p'+\frac{1}{2}q}\,.\,z-a'$ sine of $\overline{p'+\frac{1}{2}q}\,.\,z+a'$ sine of $\overline{p'+\frac{3}{2}q}\,.\,z-b'$ sine of $\overline{p'+\frac{3}{2}q}\,.\,z+b'$ sine of $\overline{p'+\frac{5}{2}q}\,.\,z-c'$ sine of $\overline{p'+\frac{5}{2}q}\,.\,z+c'$ sine of $\overline{p'+\frac{7}{2}q}\,.\,z$ &c.$=s'$.2 sine of $\frac{1}{2}qz$, put $b'-a'=a''$, $c'-b'=b''$, $d'-c'=d''$ &c. $p'-\frac{1}{2}q=p''$ and $-s'$. 2 sine of $\frac{1}{2}qz=s''$, and we have a sine of $p''z+a''$ sine of $\overline{p''+q}\,.\,z+b''$ sine of $\overline{p''+2q}\,.\,z$ &c.$=s''$, and because this is exactly similar to the original equation, (if we put a'', b'', c'', &c. for b, c, d, &c. in that, and

p'' and s'' for p and s,) it follows that if we put $b''-a''=a'''$, $c''-b''=b'''$, $d''-c''=c'''$, &c. $p''-\frac{1}{2}q=p'''$, $s''.\,2$ sine of $\frac{1}{2}qz=s'''$, that we shall have, a cos. of $p'''z+a'''$ cos. of $\overline{p'''+q}.z+b'''$, cos. of $\overline{p'''+2q}.z$ &c.$=s'''$, which is exactly similar to the second equation; (if a''', b''', c''', &c. p''' and s''' be written for a', b', c', &c. p' and s' in that,) and therefore putting $b'''-a'''=a^{\text{iv}}$, $c'''-b'''=b^{\text{iv}}$, $d'''-c'''=c^{\text{iv}}$ &c. $p'''-\frac{1}{2}q=p^{\text{iv}}$, $-s'''.\,2$ sine of $\frac{1}{2}qz$ $=s^{\text{iv}}$, we get a sine of $p^{\text{iv}}z+a^{\text{iv}}$ sine of $\overline{p^{\text{iv}}+q}\,.\,z+b^{\text{iv}}$. sine of $\overline{p^{\text{iv}}+2q}\,.\,z$, &c.$=s^{\text{iv}}$, again, similar to the first, by putting a^{iv}, b^{iv}, c^{iv}, &c. p^{iv}, s^{iv} in that equation for b, c, d, &c. p and s, and thus do we continually get equations in form similar to the first and second equations QED.

Cor. I. $s''=-s'.\,2$ sine of $\frac{1}{2}qz=-s\,.\,\overline{2\text{ sine of }\frac{1}{2}qz}|^2$, $s'''=s''$. 2 sine of $\frac{1}{2}qz=-s'\,.\,\overline{2\text{ sine of }\frac{1}{2}qz}|^2=-s\,.\,\overline{2\text{ sine of }\frac{1}{2}qz}|^3$, $s^{\text{iv}}=-s'''.\,2$ sine of $\frac{1}{2}qz=s'.\,\overline{2\text{ sine of }\frac{1}{2}qz}|^3=s\,.\,\overline{2\text{ sine of }\frac{1}{2}qz}|^4$, and in general put $s^{(\pi)}$ to represent the πth successive value of s, and we shall have $s^{(\pi)}=\pm s'.\,\overline{2\text{ sine of }\frac{1}{2}qz}|^{\pi-1}=\pm s\,.\,\overline{2\text{ sine of }\frac{1}{2}qz}|^{\pi}$, the upper sign to be taken when π being divided by 4 leaves 0 or 1, the under when it leaves 2 or 3. πth successive value of $p=p-\pi\,.\,\frac{1}{2}q$, note the values s', s'', s''', &c. I call successive sums of s, and $s=\pm\dfrac{s^{(\pi)}}{\overline{2\text{ sine of }\frac{1}{2}qz}|^{\pi}}=\pm\dfrac{s^{(\pi-1)}}{\overline{2\text{ sine of }\frac{1}{2}qz}|^{\pi-1}}$.

Corollory II. If A, B, C, &c. A′, B′, C′, &c. A″, B″, C″, &c. &c. be put for the series of the 1st, 2d, 3d differences &c. of the series a, b, c, &c. taken according to the common method of differences, we shall have the series

$a,\ a',\ b',\ c',$ &c. the same as the series $a,$ A, B, C, D, &c.
$a,\ a'',\ b'',\ c'',$ &c. - - - $a,\ a'',$ A$'$, B$'$, C$'$, &c.
$a,\ a''',\ b''',\ c''',$ &c. - - - $a,\ a''',\ b''',$ A$''$, B$''$, &c.
$a,\ a^{iv},\ b^{iv},\ c^{iv},$ &c. - - - $a,\ a^{iv},\ b^{iv},\ c^{iv},$ A$'''$, &c.
&c. &c. &c. &c. &c. &c. &c. &c. &c. &c. &c.

This is evident by taking the differences by both methods, and comparing them.

Cor. III. Likewise if A, B, C, &c. A$'$, B$'$, C$'$, &c. A$''$, B$''$, C$''$, &c. &c. be put for the series of the 1st, 2d, 3d, &c. differences of the series $a,\ a',\ b',\ c',$ &c. found by the common method of differences, then shall the series

$a,\ a'',\ b'',\ c'',\ d'',$ &c. $=$ $a,$ A, B, C, &c.
$a,\ a''',\ b''',\ c''',\ d''',$ &c. $=$ $a,\ a''',$ A$'$, B$'$, &c.
$a,\ a^{iv},\ b^{iv},\ c^{iv},\ d^{iv},$ &c. $=$ $a,\ a^{iv},\ b^{iv},$ A$''$, &c.
&c. &c. &c. &c. &c. &c. &c. &c. &c. &c. &c.

These things being known, we shall now propose some examples of their use.

Example 1. Required the sum of the infinite series sine of $pz +$ sine of $\overline{p+q}.z +$ sine of $\overline{p+2q}z +$ sine of $\overline{p+3q}.z$ &c. Here $a, b, c,$ &c.$=1, 1, 1, 1, 1,$ &c. and $a, a', b',$&c.$=1, 0, 0, 0, 0,$ &c. therefore s' or $s.2$ sine of $\frac{1}{2}qz =$ cos. of $p'z =$ cos. of $\overline{p-\frac{1}{2}q}z$ $\therefore$ the sum $s = \frac{\text{cos. of } \overline{p-\frac{1}{2}q}.z}{2 \text{ sine of } \frac{1}{2}qz}$.

Cor. I. If p and q were each $=1$, we should have, sine of $z +$ sine of $2z +$ sine of $3z$ &c. $= \frac{\text{cos. of } \frac{1}{2}z}{2 \text{ sine of } \frac{1}{2}z} = \frac{1}{2}$ cotangent of $\frac{1}{2}z$.

Cor. II. If p were $=\frac{1}{2}q$, we should have sine of $pz +$ sine of $3pz +$ sine of $5pz$ &c. $= \frac{\text{cos. of } \overline{p-p}.z}{2 \text{ sine of } pz} = \frac{1}{2 \text{ sine of } pz} = \frac{1}{2}$ cosecant of pz.

Example 2, Required the sum of the infinite series, cos. of nz $+$cos. of $\overline{n+q}z+$cos. of $\overline{n+2q}\,.\,z$ &c.
Here writing n in the room of p' we have

$$\left.\begin{array}{l} a,\ a',\ b',\ c',\ \&c.=1,\ 1,\ 1,\ 1,\ \&c. \\ a,\ a'',\ b'',\ c'',\ \&c.=1,\ 0,\ 0,\ 0,\ \&c. \end{array}\right\}$$

therefore s'' or $-s'.2$ sine of $\frac{1}{2}qz=$ sine of $p''z=$ sine of $\overline{n-\frac{1}{2}q}.z$ $\therefore$ s' the sum $=-\frac{\text{sine of }\overline{n-\frac{1}{2}q}\,.\,z}{2\text{ sine of }qz}$.

Cor. I. If $n=\frac{1}{2}q$, we shall have cos. of $nz+$ cos. of $3nz+$ cos. of $5nz$ &c. $=-\frac{\text{sine of }\overline{n-n}.z}{2\text{ sine of }\frac{1}{2}qz}=0$.

Cor. II. If $n=q$, we shall have cos. of nz $+$ cos. of $2nz+$ cos. of $3nz$ &c.$=-\frac{\text{sine of }\frac{1}{2}qz}{2\text{ sine of }\frac{1}{2}qz}=-\frac{1}{2}$.

Example 3, Required the sum of the infinite series, sine of $nz+4$ sine of $\overline{n+q}.z+9$ sine of $\overline{n+2q}.z+16$ sine of $\overline{n+3q}.z$ &c. Here $p=n$

$$\left.\begin{array}{l} \text{and } a,\ b,\ c,\ d,\ \&c.=1,\ 4,\ 9,\ 16,\ 25,\ \&c. \\ a,\ a',\ b',\ c',\ \&c.=1,\ 3,\ 5,\ 7,\ 9,\ \&c. \\ a,\ a'',\ b'',\ c'',\ \&c.=1,\ 2,\ 2,\ 2,\ 2,\ \&c. \\ a,\ a''',\ b''',\ c''',\ \&c.=1,\ 1,\ 0,\ 0,\ 0,\ \&c. \end{array}\right\}$$

therefore s''' or $-s.\left.\overline{2\text{ sine of }\frac{1}{2}qz}\right|^3=$cos. of $p'''z+$cos. of $\overline{p'''+q}\,.z=$cos. of $\overline{n-\frac{3}{2}q}\,.z$ $+$cos of $\overline{n-\frac{1}{2}q}\,.\,z$, and therefore s the sum $=-\frac{\text{cos. of }\overline{n-\frac{3}{2}q}\,.\,z+\text{cos. of }\overline{n-\frac{1}{2}q}\,.\,z}{\left.\overline{2\text{ sine of }\frac{1}{2}qz}\right|^3}$.

Cor. I. If $n=\frac{1}{2}q$, we have, sine of $nz+4$ sine of $3nz+9$ sine of $5nz$ &c. $=-\frac{\text{cos. of }\overline{-2nz}+1}{\left.\overline{2\text{ sine of }nz}\right|^3}=-\frac{\text{cos. of }\overline{2nz}+1}{\left.\overline{2\text{ sine of }nz}\right|^3}=-\frac{\text{versed sine supplement of }2nz}{\left.\overline{2\text{ sine of }nz}\right|^3}$.

Cor. II. If $n=q$, we shall have, sine of $nz+4$ sine of $2nz$ $+9$ sine of $3nz$ &c.$=-\frac{\text{cos. of }\overline{-\frac{1}{2}nz}+\text{cos. of }\frac{1}{2}nz}{\left.\overline{2\text{ sine of }\frac{1}{2}nz}\right|^3}=-\frac{2\text{ cos. of }\frac{1}{2}nz}{\left.\overline{2\text{ sine of }\frac{1}{2}nz}\right|^3}$, because, cos. of $-nz=$cos. of $+nz$.

Scholium 1. It is evident from *Cor.* II. and III. *Theorem* I. that if the coefficients of the sines (a, b, c, &c.) or of the cosines (a, a', b', c', &c.) be such that any order of differences taken according to the common method becomes $=0$, we shall then have the corresponding value, of the successive values of s, s', s'', &c. expressed in finite terms, and we shall consequently get the value of the series sought expressed in finite terms, and likewise all the intermediate values of s', s'', s''', &c. contained between s and the said corresponding successive value of s, expressed in finite terms; hence if the values of a, b, c, &c. or of a, a', b', c', &c. be respectively equal to $\frac{r}{t}g, \frac{r\,.\,\overline{r+b}}{t\,.\,\overline{t+b}}g, \frac{r\,.\,\overline{r+b}\,.\,\overline{r+2b}}{t\,.\,\overline{t+b}\,.\,\overline{t+2b}}$ &c. r, h, t, being all affirmative values, and $r-t$ a multiple of h, we may obtain the sum of the series.

In order to prove this, I shall put r_1, r_2, r_3, &c. to represent $r+h, r+2h, r+3h$, &c. t_1, t_2, t_3, &c. for $t+h, t+2h, t+3h$, &c. then will the increment of $\frac{r\,r_1\,r_2 \ldots\ldots r_\nu}{t_\varepsilon\,t_{\varepsilon+1}\,t_{\varepsilon+2} \ldots\ldots t_{\varepsilon+\nu}} = \frac{r\,r_1 \ldots\ldots r_{\nu+1}}{t_\varepsilon \ldots\ldots t_{\varepsilon+\nu+1}} - \frac{r\,r_1 \ldots\ldots r_\nu}{t_\varepsilon \ldots\ldots t_{\varepsilon+\nu}}$ (ν being supposed the only variable quantity) $= \frac{r_{\nu+1} - t_{\varepsilon+\nu+1}}{t_\varepsilon} \times \frac{r\,.\,r_1\,.\,r_2 \ldots r_{\nu+1}}{t_{\varepsilon+1} \ldots\ldots\ldots\ldots t_{\varepsilon+\nu+1}} = \frac{r-t-\varepsilon b}{t+\varepsilon b} \times \frac{r\,r_1 \ldots r_\nu}{t_{\varepsilon+1} \ldots\ldots t_{\varepsilon+\nu+1}}$; it is likewise evident that the $\overline{\nu+1}$th term of the series proposed may be expressed by $\frac{r\,r_1\,r_2\,r_3 \ldots\ldots r_\nu}{t\,t_1\,t_2\,t_3 \ldots\ldots t_\nu}\,.\,g$, ($\nu$ being a whole positive number,) this term we will call T, therefore we have, from what has been just shown, $\overset{.}{\mathrm{T}} = \frac{r-t}{t} \cdot g \cdot \frac{\overset{r\,\ldots\,r}{\nu}}{\underset{1\qquad\nu+1}{t\,\ldots\,t}}$, $\overset{..}{\mathrm{T}} = \frac{r-t}{t} \cdot \frac{r-t-h}{t+h} \cdot g \times \frac{\overset{r\,\ldots\,r}{\nu}}{\underset{2\qquad\nu+2}{t\,\ldots\,t}}$, $\overset{...}{\mathrm{T}} = \frac{r-t}{t} \cdot \frac{r-t-h}{t+h} \cdot \frac{r-t-2h}{t+2h} \cdot g \times \frac{\overset{r\,\ldots\,r}{\nu}}{\underset{3\qquad\nu+3}{t\,\ldots\,t}}$, &c. and the εth increment or difference $= \frac{r-t}{t} \cdot \frac{r-t-h}{t+h} \cdot \frac{r-t-2h}{t+2h}$ &c. $\ldots\ldots \times \frac{r-t-\overline{\varepsilon-1}\,.\,h}{t+\overline{\varepsilon-1}\,.\,h}\, g \times \frac{\overset{r\,\ldots\,r}{\nu}}{\underset{\varepsilon\qquad\nu+\varepsilon}{t\,\ldots\,t}}$ which, it is evident, will be equal to o. If $r-t=\overline{\varepsilon-1}\,.\,h$ whatever ν may be, that is, whatever term of the εth order of difference be sought it will be found equal to o; the truth of this will be likewise evinced in particular cases by the following examples.

Example 4, Required the sum of the infinite series, $\frac{3}{1}$ sine of $pz + \frac{3.4}{1.2}$ sine of $\overline{p+q}\,.\,z + \frac{3.4.5}{1.2.3}$ sine of $\overline{p+q}\,.\,z$ &c.

Here a, b, c, d, &c. $= \frac{3}{1}, \frac{3.4}{1.2}, \frac{3.4.5}{1.2.3}, \frac{3.4.5.6}{1.2.3.4}$, &c.
a, a', b', c', &c. $= 3, \frac{2.3}{1.2}, \frac{2.3.4}{1.2.3}, \frac{2.3.4.5}{1.2.3.4}$, &c.
a, a'', b'', c'', &c. $= 3,\ 0,\ 1,\ 1,$ &c.
a, a''', b''', c''', &c. $= 3, -3,\ 1,\ 0,$ &c.

therefore s''' or $- s\,.\,\overline{2 \text{ sine of } \frac{1}{2}qz}|^3 = 3$ cos. of $p'''z - 3$ cos. of $\overline{p'''+q}\,.\,z +$ cos. of $\overline{p'''+2qz}$ $\therefore s$ the sum $= [3$ cos. of $p'''z - 3$ cos. of $\overline{p'''+q}\,.\,z +$ cos. $\overline{p'''+2q}\,.\,z] \div - \overline{2 \text{ sine of } \frac{1}{2}q\,.\,z}|^3 =$

$$\frac{3 \text{ cos. of } \overline{p-\frac{3}{2}q}\, z - 3 \text{ cos. of } \overline{p-\frac{1}{2}q}.z + \text{cos of } \overline{p+\frac{1}{2}q}.z}{-\overline{2 \text{ sine of } \frac{1}{2}qz}|^3}.$$

Note. The series might have been written thus, 3 sine of $pz + 6$ sine of $\overline{p+q}\,.\,z + 10$ sine of $\overline{p+\,qz}$. &c.

Example 5, Required the sum of the infinite series, $\frac{5.}{3}$ cos. of $nz + \frac{5.6.}{3.4}$ cos. of $\overline{n+q}.z + \frac{5.6.7.}{3.4.5}$ cos. of $\overline{n+2qz}$ &c.

Here $p' = n$, therefore,

$$\left.\begin{array}{l} a,\ a',\ b',\ c',\ \&c. = \frac{5}{3},\ \frac{5.6}{3.4},\ \frac{5.6.7}{3.4.5},\ \frac{5.6.7.8}{3.4.5.6},\ \frac{5.6.7.8.9}{3.4.5.6.7},\ \&c. \\ a,\ a'',\ b'',\ c'',\ \&c. = \frac{5}{3},\ \frac{2.5}{3.4},\ \frac{2.5.6}{3.4.5},\ \frac{2.5.6.7}{3.4.5.6},\ \frac{2.5.6.7.8}{3.4.5.6.7},\ \&c. \\ a,\ a''',\ b''',\ c''',\ \&c. = \frac{5}{3},\ \frac{-2.5}{3.4},\ \frac{2.5}{3.4.5},\ \frac{2.5.6}{3.4.5.6},\ \frac{2.5.6.7}{3.4.5.6.7},\ \&c. \\ a,\ a^{iv},\ b^{iv},\ c^{iv},\ \&c. = \frac{5}{3},\ \frac{-5}{2},\ 1,\ 0,\ 0,\ \&c. \end{array}\right\}$$

therefore s^{iv} or $s'.\overline{2 \text{ sine of } \frac{1}{2}qz}|^3 = \frac{5}{3}$ sine of $p^{iv}z - \frac{5}{2}$ sine of $\overline{p^{iv}+q}.z +$ sine of $\overline{p^{iv}+2q}.z$, and therefore s' the sum sought $= \frac{\frac{5}{3} \text{ sine of } p^{iv}z - \frac{5}{2} \text{ sine of } \overline{p^{iv}+q}.z + \text{ sine of } \overline{p^{iv}+2q}.z}{\overline{2 \text{ sine of } \frac{1}{2}qz}|^3} = [\frac{5}{3}$ sine of $\overline{n-\frac{3}{2}q}.z - \frac{5}{2}$ sine of $\overline{n-\frac{1}{2}q}.z +$ sine of $\overline{n+\frac{1}{2}q}.z] : \overline{2 \text{ sine of } \frac{1}{2}qz}|^3$.

Note. The series might have been written thus, $\frac{4.5}{1.2}$ cos. of $nz + \frac{5.6}{1.2}$ cos. of $\overline{n+q}.z + \frac{6.7}{1.2}$ cos. of $\overline{n+2q}z$ &c.

Cor. If $2n = q$, s' becomes $\frac{\frac{5}{3} \text{ sine of } \overline{-2nz} - \frac{5}{2} \text{ sine of } 0 + \text{ sine of } 2nz}{\overline{2.\text{sine of } nz}|^3} = -\frac{\frac{2}{3} \text{ sine of } 2nz}{\overline{2 \text{ sine of } nz}|^3} = -\frac{4}{3}\frac{\text{sine of } nz \,..\, \text{cos. of } nz}{\overline{2 \text{ sine of } nz}|^3} = -\frac{1}{6}\frac{\text{cos. of } nz}{\overline{\text{sine of } nz}|^2}$, for the sum of the series, $\frac{5}{3}$ cos. of $nz + \frac{5.6}{3.4}$ cos. of $3nz + \frac{5.6.7}{3.4.5}$ cos. of $5nz$ &c. or its equal, $\frac{4.5}{12}$ cos. of $nz + \frac{5.6}{12}$ cos. of $3nz + \frac{6.7}{12}$ cos. of $5nz$ &c. $\therefore \frac{-2 \text{ cos. of } nz}{\overline{\text{sine of } nz}|^2} = 4.5$ cos. of $nz + 5.6.$ cos. of $3nz$ &c.

Scholium II. It is not always necessary for the differences of the coefficients to become equal to 0 to obtain the sum of the series, as will appear by

Example 6, Required the sum of the infinite series sine of $pz + g$ sine of $\overline{p+q}.z + g^2$ sine of $\overline{p+2q}.z + g^3$ sine of $\overline{p+3q}.z$ &c.

We have therefore

$$\left.\begin{array}{l} a,\ b,\ c,\ d,\ e,\ \&c.=1,g,\quad g^2,\quad g^3,\quad g^4,\quad \&c. \\ a,\ a',\ b',\ c'\ d',\&c.=1,g-1,g\times\overline{g-1},g^2\times\overline{g-1},g^3\times\overline{g-1},\&c. \\ a,\ a'',b'',c'',d'',\&c.=1,g-2,\overline{g-1}|^2,g\,\overline{g-1}|^2,g^2.\overline{g-1}|^2,\&c. \end{array}\right\}$$

Consequently, s'' or $-s\,.\,\overline{2\text{ sine of }\frac{1}{2}qz}|^2=$ sine of $p''z+\overline{g-2}\,.$ sine of $\overline{p''+q}.z+\overline{g-1}|^2.$ sine of $\overline{p''+2q}.z+\overline{g-1}|^2.g$ sine of $\overline{p''+3q}.z$ &c.$=$sine of $\overline{p-q}.z+\overline{g-2}$.sine of $pz+\overline{g-1}|^2.$ sine of $\overline{p+q}.z+\overline{g-1}|^2.g$ sine of $\overline{p+2q}.z+\overline{g-1}|^2.g^2$ sine of $\overline{p+3q}.z$ &c. but, $s=$sine of $pz+g$ sine of $\overline{p+q}.z+g^2.$ sine of $\overline{p+2q}.z$ &c. Consequently, by multiplication, division, and transposition, $\overline{g-1}|^2$ sine of $\overline{p+q}.z+\overline{g-1}|^2.g$ sine of $\overline{p+2q}.z+\overline{g-1}|^2g^2$ sine of $\overline{p+3q}.z$ &c.$=s.\frac{\overline{g-1}|^2}{g}-\frac{\overline{g-1}|^2}{g}.$ sine of pz, consequently the above equation becomes by substitution s'' or $-s.\overline{2.\text{sine of }\frac{1}{2}qz}|^2=$sine of $\overline{p-q}.z+\overline{g-2}$ sine of $pz+\frac{\overline{g-1}|^2}{g}.s-\frac{\overline{g-1}|^2}{g}.$ sine of pz, therefore, s the sum required $=$

$$\frac{\text{sine of }\overline{p-q}.z+\dfrac{\overline{\overline{g-2}-\overline{g-1}|^2}}{g}.\text{ sine of }pz}{-\dfrac{\overline{g-1}|^2}{g}-\overline{2\text{ sine of }\frac{1}{2}qz}|^2}=\frac{-g\text{ sine of }\overline{p-q}.z+\text{sine of }pz}{g^2+1-2g\text{ cos. of }qz},$$ and

by similar means, we have the sum of the series, cos. of $pz+g$ cos. of $\overline{p+q}.z+g^2$ cos. of $\overline{p+2q}.z$ &c. $=$ $\frac{-g\text{cos. of }\overline{p-q}.z+\text{ cos. of }pz}{g^2+1-2g.\text{ cos. of }qz}$.

Scholium III. Hitherto we have been considering, a series of sines and cosines, whose terms have all the same signs; but if the terms of a series proposed were alternately positive and negative, it would be necessary to divide them into two series, the one of the positive term and the other of the negative ; in order to get the sum by *Theorem* I. But the sum of a series whose terms are alternately positive and negative, may be obtained from the sum of a similar series, whose terms are all positive by a mere substitution; thus if the sum of the series, a sine of $rz - b$ sine of $\overline{r+s}\,.\,z + c$ sine of $\overline{r+2s}\,.\,z$—&c. were required, put $rz = 180^\circ - pz$, and $sz = 180 - qz$, therefore the sine of $rz =$ sine of $\overline{180^\circ - pz} =$ sine of pz, sine of $\overline{r+s}.z =$ sine of $\overline{360^\circ - \overline{p+q}.z} = -$ sine of $\overline{p+q}.z$, sine of $\overline{r+2s}\,.\,z =$ sine $\overline{540^\circ - \overline{p+2q}\,.\,z} =$ sine of $\overline{p+2q}\,.\,z$, &c.; and consequently the sum of the series, a . sine of $rz - b$. sine of $\overline{r+s}.z + c$. sine of $\overline{r+2s}.z -$ &c. $=$ the sum of the series, a sine of $pz + b$ sine of $\overline{p+q}\,.\,z + c$ sine of $\overline{p+2q}\,.\,z$ &c. ; and by the like substitution may the sum of a series of cosines, whose terms are alternately positive and negative, be deduced from the sum of a series of cosines, whose terms are all positive: all this requires the functional values of p and q to be distinct, otherwise the substitution cannot be effected; but the said sum may be deduced at once by the following

Theorem II.

If there be a series, a. sine of $pz - b$. sine of $\overline{p+q}\,.\,z + c$. sine of $\overline{p+2q}\,.\,z - d$. sine of $\overline{p+3q}\,.\,z$ &c. $= s$, then shall

a . sine of $p'z - a'$. sine of $\overline{p'+q}.z + b'$. sine of $\overline{p'+2q}.z$ &c. $= s'$

a sine of $p''z - a''$ sine of $\overline{p''+q}.z + b''$ sine of $\overline{p''+2q}.z$ &c. $= s''$

a sine of $p'''z - a'''$ sine of $\overline{p'''+q}.z + b'''$ sine of $\overline{p'''+2q}.z$ &c. $= s'''$

&c. &c. &c. &c.

And if the series be,

a cos. of $pz - b$ cos. of $\overline{p+q} \,.\, z + c$ cos. of $\overline{p+2q} \,.\, z$ &c. $= s$,
then shall,

a cos. of $p'z - a'$ cos. of $\overline{p'+q} \,.z + b'$ cos. of $\overline{p'+2q} \,.z$ &c. $= s'$

a cos. of $p''z - a''$ cos. of $\overline{p''+q} .z + b''$ cos. of $\overline{p''+2q} .z$ &c. $= s''$

&c. &c. &c. &c.

a', a'', a''', &c. b', b'', b''', &c. c', c'', c''', &c. &c. being formed from a, b, c, d, e, &c. as in *Theorem* I. p', p'', p''', &c. likewise as in *Theorem* I. $s' = 2s$. cos. of . $\frac{1}{2}qz$, $s'' = 2s'$. cos. of . $\frac{1}{2}qz$, $s''' = 2s''$. cos. of $\frac{1}{2}qz$, &c.

First, if a sine of $pz - b$ sine of $\overline{p+q} .z + c$ sine of $\overline{p+q} .z$ &c. $= s$, by multiplying by 2 cos. of $\frac{1}{2}qz$, by lemma No. II. we shall have a. sine of $\overline{p - \frac{1}{2}q} . z + a$. sine of $\overline{p+\frac{1}{2}q} . z - b$. sine of $\overline{p+\frac{1}{2}q} .z - b$. sine of $\overline{p+\frac{3}{2}q} . z + c$. sine of $\overline{p+\frac{3}{2}q} . z$ &c. $= s$. 2 cos. of $\frac{1}{2}qz$; consequently, putting $b - a = a'$, $c - b = b'$, &c. $p - \frac{1}{2}q = p'$, $s' = 2\,s$ cos. of $\frac{1}{2}qz$, we have, a sine of $p'z - a'$ sine of $\overline{p'+q} . z + b'$ sine of $\overline{p'+2q} . z$ &c. $= s'$, which being exactly similar in form to the original series, the other series will be deduced from this by continually proceeding in the same method.

Again, if a . cos. of $pz - b$. cos. of $\overline{p+q} .z + c$. cos. of $\overline{p+2q} .z$ &c. $= s$, we have by multiplying by 2 cos. of $\frac{1}{2}qz$ by the help of lemma No. III., a cos. of $p'z - a'$ cos. of $\overline{p'+q} .z + b'$ cos. of $\overline{p'+2q} . z$ &c. $= s$, which being exactly similar in form, to the original, we may obtain the other series, which are likewise similar in form by the same mode of proceeding.

Cor. the πth successive value of $s = s . \overline{2 \text{ cos. of } \frac{1}{2}qz}|^{\pi}$, the πth successive value of $p = p - \pi . \frac{1}{2}$ and $s = \dfrac{\pi\text{th successive value of } s}{\overline{2 \text{ cos. of } \frac{1}{2}qz}|^{\pi}}$.

Example 1, Required the sum of the series, sine of $pz -$

sine of $\overline{p+q}.z+$ sine of $\overline{p+2q}z-$ &c. and likewise of, cos. of $pz-$ cos. of $\overline{p+q}.z+$ cos. of $\overline{p+2q}.z-$ &c.

Here in both, $\left.\begin{array}{l} a,\ b,\ c,\ d, \text{ \&c.} = 1,\ 1,\ 1,\ 1, \text{ \&c.} \\ a,\ a',\ b',\ c', \text{ \&c.} = 1,\ 0,\ 0,\ 0, \text{ \&c.} \end{array}\right\}$ therefore, in the first series we have, s' or $s\ .\ 2$ cos. of $\frac{1}{2}qz =$ sine of $p'z$ and $\therefore s = \frac{\text{sine of } p'z}{2 \text{ cos. of } \frac{1}{2}qz} = \frac{\text{sine of } \overline{p-\frac{1}{2}q}z}{2 \text{ cos. of } \frac{1}{2}qz}$, and for the second series we have, s', or, $s\ .\ \overline{2 \text{ cos. of } \frac{1}{2}qz} =$ cos. of $\overline{p-\frac{1}{2}q}\ .\ z$, and therefore, $s = \frac{\text{cos. of } \overline{p-\frac{1}{2}q}.z}{2 \text{ cos. of } \frac{1}{2}qz}$.

Cor. 1. If $p=q$ the first series will be, sine of $pz-$ sine of $2pz+$ sine of $3pz$ &c.$= \frac{\text{sine of } \frac{1}{2}pz}{2 \text{ cos. of } \frac{1}{2}pz} = \frac{1}{2}$ tangent of $\frac{1}{2}pz$, and the second, cos. of $pz -$ cos. of $2pz +$ cos. of $3pz$ &c. $= \frac{\text{cos. of } -\frac{1}{2}pz}{2 \text{ cos. of } \frac{1}{2}pz} = \frac{1}{2}$.

Scholium. Though we have given two theorems, the one for a series whose terms are all positive, and the other for a series whose terms are alternately positive and negative; they are both true whatever the signs of the terms be, provided that proper signs be used in the operation; that is, if any term should have a contrary sign, to the sign of that term contained in the enunciation of the theorem used, then a contrary sign must likewise be prefixed to it in the operation; thus, for instance, if for a series whose terms are all positive we should use *Theorem* II. or for a series whose terms are alternately positive and negative we should use *Theorem* I., for a, b, c, d, &c. we must write $a, -b, c, -d$, &c. and therefore

$a,\ a',\ b'$, &c.$= a, -\overline{a+b},\ \overline{c+b},\ -\overline{d+c}$, &c.$=$(suppose) $a, -a_{,}, b_{,}, -c_{,}$, &c.

$a,\ a'',\ b''$, &c.$= a, -\overline{a_{,}+a},\ \overline{b_{,}+a_{,}},\ -\overline{c_{,}+b_{,}}$, &c.$=$ - - - - $a, -a_{,,}, b_{,,}, -c_{,,}$, &c.

$a,\ a''',\ b'''$, &c.$= a, -\overline{a_{,,}+a},\ \overline{b_{,,}+a_{,,}}, -\overline{c_{,,}+b_{,,}}$, &c.$=$ - - - - - - &c.

&c. &c. &c. &c. &c. &c. &c. &c. &c. &c.

and consequently we shall get new series, the coefficients of whose terms are formed from the coefficients in the preceding series, by addition instead of subtraction; and may be of good purpose on some occasions. And if we alternately use these theorems the operation will be performed by alternately taking sums and differences; and this will amount to the same as taking the differences of the alternate terms, beginning always with two noughts: but, for the more readily comprehending this, we shall offer a theorem which moreover is the first of these theorems I discovered, but previously thereto shall propose

Example 2. Let the series be either of these, sine of $pz+r$ sine of $\overline{p+q}.z+r.\frac{r+1}{2}$ sine of $\overline{p+2q}.z+r.\frac{r+1}{2}.\frac{r+2}{3}$ sine of $\overline{p+3q}.z+$ &c., cos. of $pz+r$ cos. of $\overline{p+q}.z+r.\frac{r+1}{2}$. cos. of $\overline{p+2q}.z+$ &c., sine of $pz-r$ sine of $\overline{p+q}.z+r.\frac{r+1}{2}$ sine of $\overline{p+2q}.z-$ &c. or, cos. of $pz-r$ cos. of $\overline{p+q}.z+r.\frac{r+1}{2}$ cos. of $\overline{p+2q}.z-$ &c. r being a whole positive number, the terms in the two first series all positive, and in the two last alternately positive and negative.

The coefficients being,

Of 1st term.	2d.	3d.	4th.	5th.	
$1,$	$r,$	$r.\frac{r+1}{2},$	$r.\frac{r+1}{2}.\frac{r+2}{3},$	$r.\frac{r+1}{2}.\frac{r+2}{3}.\frac{r+3}{4},$	&c.

the first differences

$$1,\frac{r-1}{1},\frac{r-1}{1}.\frac{r}{2},\frac{r-1}{1}.\frac{r}{2}.\frac{r+1}{3},\frac{r-1}{1}.\frac{r}{2}.\frac{r+1}{3}.\frac{r+2}{4},\text{ \&c.}$$

2d differences

$$1,\frac{r-2}{1},\frac{r-2}{1}.\frac{r-1}{2},\frac{r-2}{1}.\frac{r-1}{2}.\frac{r}{3},\frac{r-2}{1}.\frac{r-1}{2}.\frac{r}{3}.\frac{r+1}{4},\text{ \&c.}$$

3d differences

Of 1st term. 2d. 3d. 4th. 5th.

$$1,\ \frac{r-3}{1},\ \frac{r-3}{1}\cdot\frac{r-2}{2},\ \frac{r-3}{1}\cdot\frac{r-2}{2}\cdot\frac{r-1}{3},\ \frac{r-3}{1}\cdot\frac{r-2}{2}\cdot\frac{r-1}{3}\cdot\frac{r}{4},\ \text{\&c.}$$

And in general, πth differences

$1,\ \frac{r-\pi}{1},\ \frac{r-\pi}{1}\cdot\frac{r-\pi-1}{2},\ \frac{r-\pi}{1}\cdot\frac{r-\pi-1}{1}\cdot\frac{r-\pi-2}{3}$, &c. to be continued to $\overline{\pi+1}$ terms, and the remaining terms will be the $\overline{\pi+1}$th term multiplied by $\frac{r}{\pi+1},\ \frac{r}{\pi+1}\cdot\frac{r+1}{\pi+2},\ \frac{r}{\pi+1}\cdot\frac{r+1}{\pi+2}\cdot\frac{r+2}{\pi+3}$, &c. and consequently if π be $= r$, all the terms of the πth differences except the first will vanish. Hence we have by *Theorem* I. and its *Cor.* 1. the sum of the series, sine of $pz+r$ sine of $\overline{p+q}.z+r.\frac{r+1}{2}$. sine of $\overline{p+2q}.z+$ &c. $=\pm\frac{\text{sine of }\overline{p-\frac{1}{2}rq}.z}{\overline{2\text{ sine of }\frac{1}{2}qz}|^r}$, if r be even, but $\pm\frac{\text{cos. of }\overline{p-\frac{1}{2}rq}.z}{\overline{2\text{ sine of }\frac{1}{2}qz}|^r}$, if r be odd, the upper signs to be taken when r being divided by 4 leaves 0 or 1, and the under signs when it leaves 2 or 3. And the sum of the series, cos. of $pz+r$ cos. of $\overline{p+q}.z+r.\frac{r+1}{2}$ cos. of $\overline{p+2q}.z+$, &c. $=\pm\frac{\text{sine of }\overline{p-\frac{1}{2}rq}.z}{\overline{2\text{ sine of }\frac{1}{2}qz}|^r}$ if r be odd, but $\pm\frac{\text{cos. of }\overline{p-\frac{1}{2}rq}.z}{\overline{2\text{ sine of }\frac{1}{2}qz}|^r}$ if even, the upper signs to be taken, if r leaves 3 or 0 when divided by 4, and the under if it should leave 2 or 1. In deducing the sum of this series from the said *Cor.* it is necessary to put $\overline{p+\frac{1}{2}q}$ for p and $r+1$ for the π used there. The sum of the series, sine of $pz-r$ sine of $\overline{p+q}.z+r.\frac{r+1}{2}$ sine of $\overline{p+2q}.z-$, &c. by *Theorem* II. is $=\frac{\text{sine of }\overline{p-\frac{1}{2}rq}.z}{\overline{2\text{ cos. of }\frac{1}{2}qz}|^r}$: and the sum of the series, cos. of $pz-r$ cos. of $\overline{p+q}.z+r.\frac{r+1}{2}$ cos. of $\overline{p+2q}.z-$ &c. by the same $=\frac{\text{cos. of }\overline{p-\frac{1}{2}rq}.z}{\overline{2\text{ cos. of }\frac{1}{2}q}|^r}$.

Corollary. Because sine of $pz = pz - \frac{p^3 z^3}{2.3} + \frac{p^5 z^5}{2.3.4.5}$, &c. sine of $\overline{p+q}z = \overline{p+q}\,.\,z - \frac{\overline{p+q}|^3}{2.3}\,.\,z^3 + \frac{\overline{p+q}|^5}{2.3.4.5}\,z^5$, &c. sine of $\overline{p+2q}.z = \overline{p+2q}\,.\,z - \frac{\overline{p+2q}|^3}{2.3}\,.\,z^3 + \frac{\overline{p+2q}|^5}{2.3.4.5}\,.\,z^5$, &c. &c., and $\frac{1}{\overline{\text{cos. of } \frac{1}{2}qz}|^r} = \frac{1}{\overline{1 - \frac{\frac{1}{4}q^2 z^2}{1.2} + \frac{\frac{1}{16}q^4 z^4}{1.2.3.4} \text{ \&c.}}|^r} = 1 + Az^2 + Bz^4 + Cz^6$, &c.

Where A, B, C, &c. stand for the coefficients of the multinomial, $1 - \frac{\frac{1}{4}q^2 z^2}{1.2} + \frac{\frac{1}{16}q^4 z^4}{1.2.3.4}$ &c. raised to the $-r$ power, and consequently r only concerned in them by pure powers; hence this being multiplied by $\overline{p-\frac{1}{2}qr}.z - \frac{\overline{p-\frac{1}{2}qr}|^3.z^3}{2.3} + \frac{\overline{p-\frac{1}{2}qr}|^5.z^5}{2.3.4.5}$ &c. the value of sine of $\overline{p-\frac{1}{2}qr}.z$, we obtain from the equation sine of $pz - r\,.$ sine of $\overline{p+q}\,.\,z + r\,.\,\frac{r+1}{2}$ sine of $\overline{p+2q}\,.\,z$, &c. $= \frac{\text{sine of } \overline{p-\frac{1}{2}qr}.z}{2 \text{ cos. of } \overline{\frac{1}{2}qz}|^r}$, the series $\overline{p - r.\overline{p+q} + r.\frac{r+1}{2}.\overline{p+2q} - \text{\&c.}}|.z$ $- \overline{p^3 - r.\overline{p+q}|^3 + r.\frac{r+1}{2}\overline{p+2q}|^3 - \text{\&c.}}|.\frac{z^3}{2.3} + \overline{p^5 - r.\overline{p+q}|^5 + r\,\frac{r+1}{2}.\overline{p+2q}|^5 - \text{\&c.}}|\,.\frac{z^5}{2.3.4.5}$, &c. $= \frac{p-\frac{1}{2}qr}{2^r}z - \overline{\frac{\overline{p-\frac{1}{2}qr}|^3}{2^r} - \frac{2.3A}{2^r}.\overline{p-\frac{1}{2}qr}}|\,.\frac{z^3}{2.3} + \overline{\frac{\overline{p-\frac{1}{2}qr}|^5}{2^r} - \frac{4.5A}{2^r}.\overline{p-\frac{1}{2}qr}|^3 + \frac{B}{2^r}.\overline{p-\frac{1}{2}qr}}|.\frac{z^5}{2.3.4.5}$ &c., the law of continuation being evident in both series, consequently by comparing the homologous terms we obtain the sum of the series, $p - r.\overline{p+q} + r.\frac{r+1}{2}.\overline{p+2q} - r.\frac{r+1}{2}\,\frac{r+2}{3}.\overline{p+3q}$ &c. $= \frac{p-\frac{1}{2}qr}{2^r}$, of $p^3 - r.\overline{p+q}|^3 + r.\frac{r+1}{2}.\overline{p+2q}|^3$ &c. $= \frac{\overline{p-\frac{1}{2}qr}|^3}{2^r} - \frac{2.3}{2^r}.\overline{p-\frac{1}{2}qr}$, of the series $p^5 - r.\overline{p+q}|^5 + r.\frac{r+1}{2}.\overline{p+2q}|^5 -$ &c. $= \frac{\overline{p-\frac{1}{2}q}|^5}{2^r} - \frac{4.5.A}{2}.\overline{p-\frac{1}{2}qr}|^3 + \frac{2.3.4.5\,B}{2^r}.\overline{p-\frac{1}{2}qr}$, and so for the other odd powers, r being only concerned in these expressions by pure powers, and by similar means we may from the equation, cos. of $pz - r.\text{cos. of }\overline{p+q}.z + r.\frac{r+1}{2}\text{cos. of }\overline{p+2q}.z$ &c. $= \frac{\text{cos. of }\overline{p-\frac{1}{2}qrz}}{2\text{ cos. of }\overline{\frac{1}{2}qz}|^r}$, obtain the series $\overline{1-r+r.\frac{r+1}{2}-r.\frac{r+1}{2}.\frac{r+2}{3}\text{ \&c.}} - \overline{p^2-r.\overline{p+q}|^2+r.\frac{r+1}{2}.\overline{p+2q}|^2-\text{\&c.}}\,\frac{z^2}{1.2} + \overline{p^4-r.\overline{p+q}|^4+r.\frac{r+1}{2}.\overline{p+2q}|^4-\text{\&c.}}\,.\frac{z^4}{1.2.3.4} - \text{\&c.} = \frac{1}{2^r} - \frac{\overline{p-\frac{1}{2}qr}|^2-1.2.A}{2^r}.\frac{z^2}{1.2} + \frac{\overline{p-\frac{1}{2}qr}|^4-3.4.A.\overline{p-\frac{1}{2}qr}|^2+1.2.3.4.B}{2^r}.\frac{z^4}{1.2.3.4}$; and consequently by again comparing the homologous terms, we find $1-r+r.\frac{r+1}{2}-r.\frac{r+1}{2}.\frac{r+2}{3}$ &c. $=\frac{1}{2^r}$, as it is well known to be, $p^2-r.\overline{p+q}|^2+r.\frac{r+1}{2}\overline{p+2q}|^2-$ &c. $=\frac{\overline{p-\frac{1}{2}qr}|^2-1.2.A}{2^r}$, $p^4-r.\overline{p+q}|^4+r.\frac{r+1}{2}\overline{p+2q}|^4$ &c. $=\frac{\overline{p-\frac{1}{2}qr}|^4-3.4\,A\,\overline{p-\frac{1}{2}qr}|^2+1.2\,3.4.B}{2^r}$, and so for the other even powers, r being only concerned in these expressions by pure powers.

Hence r being a whole positive number, the sum of the series, $p^m-r.\overline{p+q}|^m+r.\frac{r+1}{2}.\overline{p+2q}|^m-r.\frac{r+1}{2}.\frac{r+2}{3}.\overline{p+3q}|^m$ &c, m likewise being a whole positive number, may be always expressed by $\frac{1}{2^r}\times$ by a series of finite terms of pure powers of r whose coefficients are given, of the form $a+br+cr^2$ &c. p, q, and m being given values, and a, b, c, &c. determinate values independent of r; merely by comparing the coefficients of the homologous powers, of z, in the two equations of the series above. Now if we can prove that the same expressions, derived from the comparison of the coefficients of the homologous powers of z, give the sum of the series $p^m-r.\overline{p+q}|^m+r.\frac{r+1}{2}.\overline{p+2q}^m$ &c. whether r be a whole positive number or not, it will follow that the series, sine of $pz - r$ sine of $\overline{p+q}.z + r.\frac{r+1}{2}$ sine of $\overline{p+2q}.z$ &c. will be equal to $\frac{\text{sine of } \overline{p-\frac{1}{2}qr.z}}{\overline{2 \text{ cos. of } \frac{1}{2}qz}|^{r}}$ and the series, cos. of $pz - r$ cos. of $\overline{p+q}.z + r.\frac{r+1}{2}$ cos. of $\overline{p+2q}.z$ &c. $= \frac{\text{cos. of } \overline{p-\frac{1}{2}qr.z}}{\overline{2 \text{ cos. of } \frac{1}{2}qz}|^{r}}$, whether r be a whole positive number or not.

And in order to prove this requisite, we shall first premise that if we have the sum of the series $p^m - r.\overline{p+q}|^m + r.\frac{r+1}{2}.\overline{p+2q}|^m - r.\frac{r+1}{2}.\frac{r+2}{3}.\overline{p+3q}|^m$ &c. whatever r may be, (m, p and q being given quantities) expressed by a series, $\frac{1}{2^r} \times \overline{A+Br+Cr^2}$ &c. of finite terms in which the *functional* values of p and r are distinct, A, B, C, &c. being given quantities independent of r, we may likewise find the sum of the series $p^{m+1} - r.\overline{p+q}|^{m+1} + r.\frac{r+1}{2}\overline{p+2q}|^{m+1}$ &c. for this series is equal to $p.p^m - r.\overline{p+q}.\overline{p+q}|^m + r.\frac{r+1}{2}.\overline{p+2q}.\overline{p+2q}|^m$ &c. $= p \times \overline{p^m - r.\overline{p+q}|^m + r.\frac{r+1}{2}.\overline{p+2q}|^m \text{ \&c.}} - rq \times \overline{\overline{p+q}|^m - \overline{r+1}\,\overline{p+2q}|^m + \overline{r+1}\,\frac{r+2}{2}\,\overline{p+3q}|^m - \overline{r+1}.\frac{r+2}{2}.\frac{r+3}{3}.\overline{p+4q}|^m}$ &c. but $p^m - r.\overline{p+q}|^m + r.\frac{r+1}{2}.\overline{p+2q}|^m$ &c. is equal to $\frac{1}{2^r} \times \overline{A+Br+Cr^2 \text{ \&c.}}$ and if in this we write $\overline{r+1}$ for r and $\overline{p+q}$ for p, we shall have the sum of the series $\overline{p+q}|^m - \overline{r+1}.\overline{p+2q}|^m + \overline{r+1}\,\frac{r+2}{2}.\overline{p+3q}|^m$ &c. $= \frac{1}{2^{r+1}} \times \overline{A_{,} + B_{,}.\overline{r+1} + C_{,}.\overline{r+1}|^2}$ &c, $A_{,}$, $B_{,}$, $C_{,}$, &c. standing for the values that A, B, C, &c. become by writing $\overline{p+q}$ for p; and this may evidently be reduced to an expression of the finite terms of the form $\frac{1}{2^r} \times \overline{A'+B'r+C'r^2 \text{ \&c.}}$ and consequently will the sum of the series $p^{m+1} - r . \overline{p+q}|^{m+1} + r . \frac{r+1}{2} . \overline{p+2q}|^{m+1}$ &c. be = the expression $\frac{1}{2^r} \times \overline{pA + pBr + pCr^2 \text{ \&c.} - qA'r - qB'r^2 \text{ \&c.}}$ of finite terms. This being proved, it follows, because the sum of the series $p^m - r . \overline{p+q}|^m + r . \frac{r+1}{2} . \overline{p+2q}|^m$ &c. when m is equal to o, is equal to $1 - r + r . \frac{r+1}{2}$ &c. =, by the binomial theorem, $\frac{1}{2^r}$, whatever r may be, that the sum of the series $p - r . \overline{p+q} + r . \frac{\overline{r+1}}{2} \overline{p+2q}$ &c. namely, the said series when m is equal to 1, may be expressed by $\frac{1}{2^r} \times$ a series of pure powers of r of a finite number of terms whatever r may be, and comes out by the bye $\frac{1}{2^r} \times \overline{p - \frac{1}{2}qr}$, the same as above, and consequently by writing 1, 2, 3, 4, 5, 6, &c. one after the other for, m, we shall find that the sum of the series $p^m - r . \overline{p+q}|^m + r . \frac{r+1}{2} \overline{p+2q}|^m$ &c. may always be expressed by $\frac{1}{2^r}$ multiplied by a series of finite terms in the form $A + Br + Cr^2$ &c. A, B, C, &c. p, q, m, &c. being independent of r; and m a whole positive number. And these will, we shall prove without running through all the infinite cases, be the very same expressions as those given above, by comparing the coefficients of the homologous powers of z. In order to this we observe, since we have just proved that the sum of the said series, whatever r may be, may be expressed by $\frac{1}{2^r} \times$ series $\overline{A + Br + Cr^2}$ &c. of a finite number of terms, and from the comparison of the homologous powers, that when r is a whole number it may be expressed by $\frac{1}{2^r} \times$ value $a + br + cr^2$ &c. of a finite number of terms, it follows that when r is any whole number, that these two values must then be equal to each other, and ∴ that $A+Br^2+Cr^2$ &c. containing a finite number of terms must be then equal to $a+br+cr^2$ &c. containing a finite number of terms, and consequently the highest power of r and its coefficient must be the same in both series, otherwise by increasing r by the same in both, one side of the equation would become greater than the other, which is absurd; consequently the highest power of r and its coefficient is the same in both, and will destroy each other, and consequently the next highest powers of r and likewise their coefficients must be the same with each other, and will therefore be destroyed, &c. Hence the powers of r and their respective coefficients being the same in both, the expressions themselves must be the same in every respect, whether r be a whole number or not.

Hence we have not only given two different means of summing the series $p^m - r . \overline{p+q}|^m$ &c. (m being a whole positive number) whatever r may be, which indeed was not our chief object, but we have likewise proved that the series sine of $pz - r$ sine of $\overline{p+q}.z$ &c. $= \dfrac{\text{sine of } \overline{p-\frac{1}{2}qr}.z}{\overline{2 \text{ cos. of } \frac{1}{2}qz}|^r}$, and the series cos. of $pz - r$ cos. of $\overline{p+q}.z$ &c. $= \dfrac{\text{cos. of } \overline{p-\frac{1}{2}qr}.z}{\overline{2 \text{ cos. of } \frac{1}{2}qz}|^r}$, whatever r may be, the same as LANDEN finds.

Cor. II. Because these two series are equally true, whatever p may be, if for p we write $qr-p$ throughout, in the first we shall have, sine of $\overline{qr-p} . z - r$ sine of $\overline{\overline{r+1} . q-p} . z + r . \frac{r+1}{2}$ sine of $\overline{\overline{r+2}.q-p}.z$ &c. $= \dfrac{\text{sine of } \overline{\frac{1}{2}qr-p}.z}{\overline{2 \text{ cos. of } \frac{1}{2}qz}|^r} = -\dfrac{\text{sine of } \overline{p-\frac{1}{2}q\,r}\,z}{\overline{2 \text{ cos. of } \frac{1}{2}q}|^r}$.

Consequently sine of $\overline{qr-p}.z - r$ sine of $\overline{\overline{r+1}q-p}.z +$ &c.

$= -$ sine of $pz + r$ sine of $\overline{p+q}\,.\,z -$, &c. and from cos. of $pz - r$ cos. of $\overline{p+q}\,.\,z$, &c. $= \frac{\text{cos. of } \overline{p - \frac{1}{2}qr}\,.\,z}{2 \text{ cos. of } \frac{1}{2}qz|^{r}}$ by the like substitution we get cos. of $\overline{qr-p}\,.\,z - r$ cos. of $\overline{\overline{r+1}\,.\,q - p}\,.\,z + r\,.$ $\frac{r+1}{2}$ cos. of $\overline{\overline{r+2}\,.\,q - p}\,.\,z$ &c. $= \left(\frac{\text{cos. of } \overline{\frac{1}{2}qr - p}\,.\,z}{2 \text{ cos. of } \frac{1}{2}qz|^{r}} = \frac{\text{cos. of } \overline{p - \frac{1}{2}qr}\,.\,z}{2 \text{ cos. of } \frac{1}{2}qz|^{r}} =\right)$ cos. of $pz - r$ cos. of $\overline{p+q}\,.\,z + r\,.\,\frac{r+1}{2}$ cos. of $\overline{p+2q}\,.\,z$ &c. and these by *Cor.* I. are true, whatever r may be.

Cor. III. If p be $= \frac{1}{2}qr$ we shall have, cos. of $\frac{1}{2}qrz - r$ cos. of $\frac{1}{2}q\,.\,\overline{r+2}\,.\,z + r\,.\,\frac{r+1}{2}\,.$ cos. of $\frac{1}{2}q\,.\,\overline{r+4}\,.\,z$ &c. $= \frac{\text{cos. of } 0}{2 \text{ cos. of } \frac{1}{2}qz|^{r}}$ $= \frac{1}{2 \text{ cos. of } \frac{1}{2}qz|^{r}}$, or if A be written for $\frac{1}{2}qz$ we shall have, cos. of rA $- r$ cos. of $\overline{r+2}$ A $+ r\,.\,\frac{r+1}{2}$ cos. of $\overline{r+4}\,.$ A $-$ &c. $= \frac{1}{2 \text{ cos. of A}|^{r}}$ which is the same in substance as SIMPSON's lemma, page 67 of his Tracts.

Cor. IV. If we put $pz = 180° - tz$, $qz = 180° - sz$ we shall have according to *Scholium* III. at the end of the examples to *Theorem* I.

sine of $tz + r$ sine of $\overline{t+s}\,.\,z + r\,.\,\frac{r+1}{2}$ sine of $\overline{t+2s}\,.\,z$ &c. $= \frac{\text{sine of } \overline{180° - tz - \frac{1}{2}r\,.\,\overline{180° - sz}}}{2 \text{ cos. of } \overline{90° - \frac{1}{2}qz}|^{r}} = \frac{\text{sine of } 90°\,.\,\overline{r + \overline{t - \frac{1}{2}rs}\,.\,z}}{2 \text{ sine of } \frac{1}{2}sz|^{r}}$ and $-$ cos. of $tz - r$ cos. of $\overline{t+s}\,.\,z - r\,.\,\frac{r+1}{2}$ cos. of $\overline{t+2s}\,.\,z$ &c. $=$ $\frac{\text{cos. of } \overline{180° - tz - \frac{1}{2}r\,.\,\overline{180° - sz}}}{2 \text{ cos. of } \overline{90° - \frac{1}{2}sz}|^{r}} = -\frac{\text{cos. of } \overline{90° + \overline{t - \frac{1}{2}rs}\,.\,z}}{2 \text{ of sine } \frac{1}{2}sz|^{r}}$ $\therefore$ cos. of $tz + r$ cos. of $\overline{t+s}\,.\,z + r\,.\,\frac{r+1}{2}$ cos. of $\overline{t+2s}\,.\,z$ &c. $= \frac{\text{cos. of } \overline{90°\; r + \overline{t - \frac{1}{2}rs}\,.\,z}}{2 \text{ sine of } \frac{1}{2}sz|^{r}}$ and if in these r be a whole number, and p and q be written for t and s, we shall have the results determined above. Many more corollaries may be derived from these.

Theorem III.

If there be formed a series of terms

$$\begin{array}{ccccccc} a, & b, & c, & d, & e, & f, & \&c. \\ a, & b, & a', & b', & c', & d', & \&c. \\ a, & b, & a'', & b'', & c'', & d'', & \&c. \\ a, & b, & a''', & b''', & c''', & d''', & \&c. \\ \&c, & \&c, & \&c, & \&c, & \&c, & \&c, & \&c. \end{array}$$

The terms of each series being formed from those immediately above, by taking the alternate differences of the terms, always beginning with o, o; that is, taking o from the 1st term, o from the 2d term, 1st term from 3d term, 2d term from 4th term, &c. in any of the series, for 1st, 2d, 3d, &c. terms of the next series. And p' be put $= p - q$, $p'' = p' - q$, $p''' = p'' - q$, &c. $s' = s \,.\, 2$ sine of qz, $s'' = -s' \,.\, 2$ sine of qz, $s''' = s'' \,.\, 2$ sine of qz, $s^{iv} = -s''' \,.\, 2$ sine of qz, &c. I say if there be a series a sine of $pz + b$ sine of $\overline{p+q} \,.\, z + c$ sine of $\overline{p+2q} \,.\, z + d$ sine of $\overline{p+3q} \,.\, z$, &c. $= s$, we shall have,

a cos. of $p'z + b$ cos. of $\overline{p'+q} \,.\, z + a'$ cos. of $\overline{p'+2q} \,.\, z + b'$ cos. of $\overline{p+3q} \,.\, z$, &c. $= s'$

a sine of $p''z + b$ sine of $\overline{p''+q} \,.\, z + a''$ sine of $\overline{p''+2q} \,.\, z + b''$ sine of $\overline{p+3q} \,.\, z$, &c. $= s''$, &c.

For multiplying the first of these by 2 sine of qz, by help of lemma No. I. we shall have, $a \,.$ cos. of $\overline{p-q} \,.\, z - a \,.$ cos. of $\overline{p+q} \,.\, z + b \,.$ cos. of $pz - b \,.$ cos. of $\overline{p+2q} \,.\, z + c$ cos. of $\overline{p+q} \,.\, z - c$ cos. of $\overline{p+3q} \,.\, z + d$ cos. of $\overline{p+2q} \,.\, z - d$ cos. of $\overline{p+4q}$

$\cdot z + e$ cos. of $\overline{p+3q}\,.\,z$, &c. $= s\,.\,2$ sine of qz, therefore putting $a' = c - a$, $b' = d - b$, $c' = e - c$, &c. $p' = p - q$, $s' = s\,.\,2$ sine of qz, we get a cos. of $p'z + b$ cos. of $\overline{p'+q}\,.\,z + a'$ cos. of $\overline{p'+2q}\,.\,z + b'$ cos. of $\overline{p+3q}\,.\,z$, &c. $= s'$, and multiplying this by 2 sine of qz, by help of lemma No. II. we get $a\,.$ sine of $\overline{p'+q}\,z$ $-\,a$ sine of $\overline{p'-q}\,.\,z + b$ sine of $\overline{p'+2q}\,.\,z - b$ sine of $p'z + a'$. sine of $\overline{p'+3q}\,.\,z - a'$ sine of $\overline{p'+q}\,.\,z + b'$ sine of $\overline{p'+4q}\,.\,z$ $-\,b'$. sine of $\overline{p'+2q}\,.\,z$ &c. $= s'\,.\,2$ sine of qz, therefore putting $a' - a = a''$, $b' - b = b''$, $c' - a' = c''$ &c. $p' - q = p''$, $-s'$. 2 sine of $qz = s''$ we have, $a\,.$ sine of $p''z + b\,.$ sine of $\overline{p''+q}\,.\,z$ $+\,a''$. sine of $\overline{p''+2q}\,.z + b''$. sine of $\overline{p''+3q}\,.z$ &c. $= s''$, which being exactly similar in form to the original series, the successive series, which will be of a similar form to the second or first of the series, will be deduced by the like operations and substitutions. Q. E. D.

Corollary I. The πth successive value of s is $= \pm s\,.\,\overline{2 \text{ sine of } qz}|^{\pi}$ or $\pm s'\,.\,\overline{2 \text{ sine of } qz}|^{\pi-1}$, the upper sign to be taken when π being divided by 4 leaves 0 or 1, otherwise the under sign and the πth successive value of $p = p - \pi\,.q$.

Corollary II. These operations are performed by differences whether the signs be all positive, or alternately positive and negative.

Example 1. Required the sum of the series n sine of $pz +$ $\overline{n+r}$ sine of $\overline{p+q}\,.\,z + \overline{n+2r}$ sine of $\overline{p+2q}\,.\,z$ &c.

$$\left.\begin{array}{llllllll} \text{Here } a, b, c, d, & \text{\&c.} = n, & n+r, & n+2r, & n+3r, & n+4r, & \text{\&c.} \\ a, b, a', b', & \text{\&c.} = n, & n+r, & 2r, & 2r, & 2r, & \text{\&c.} \\ a, b, a'', b'', & \text{\&c.} = n, & n+r, & 2r-n, & r-n, & 0, & \text{\&c.} \end{array}\right\}$$

$\therefore s'' = -s\,.\,\overline{2 \text{ sine of } qz}|^{2} = n$ sine of $p''z + \overline{n+r}$ sine of $\overline{p''+q}$

$. z + \overline{2r-n}$ sine of $\overline{p''+2q} . z + \overline{r-n}$ sine of $\overline{p''+3q} . z$ ∴ by restoration and division we have s the sum $= [n$ sine of $\overline{p-2q} . z + \overline{n+r}$ sine of $\overline{p-q} . z + \overline{2r-n}$ sine of $pz + \overline{r-n}$ sine of $\overline{p+q} . z] \div -\overline{2 \text{ sine of } qz}|^2$: had we used *Theorem* I. we should have gotten a more simple valuation; namely, $s = \frac{n \text{ sine of } \overline{p-q} . z + \overline{r-n} \text{ sine of } pz}{-\overline{2 \text{ sine of } \frac{1}{2}qz}|^2}$ which is reducible to the other by multiplying the upper and under terms by $\overline{2 \text{ cos. of } \frac{1}{2}qz}|^2$ by help of lemma No. II. and III. Had the terms been alternate positive and negative we should have had

$$\left.\begin{array}{l} a, b, c, d, e, \&c. = n, -\overline{n+r}, \ \overline{n+2r}, -\overline{n+3r}, \ \overline{n+4r}, \&c. \\ a, b, a', b', c', \&c. = n, -\overline{n+r}, + 2r, \ - 2r, \ + 2r, \ \&c. \\ a, b, a'', b'', c'', \&c. = n, -\overline{n+r}, \ \overline{2r-n}, -\overline{r-n}, \ \ 0, \ \&c. \end{array}\right\}$$

and therefore $s = [n$ sine of $\overline{p-2q} . z - \overline{n+r}$ sine of $\overline{p-q} . z + \overline{2r-n}$ sine of $pz - \overline{r-n}$ sine of $\overline{p+q} . z] \div -\overline{2 \text{ sine of } qz}|^2$. If we had used *Theorem* II. we should have obtained $s = \frac{n \text{ sine of } \overline{p-q} . z - \overline{r-n} . \text{ sine of } pz}{\overline{2 \text{ cos. of } \frac{1}{2}qz}|^2}$ which is reducible to the other by multiplying the upper and under terms by $\overline{2 \text{ sine of } \frac{1}{2}qz}|^2$, by help of lemma No. I. and II.

Theorem IV.

If there be a series, a . sine of $pz + b$. sine of $\overline{p+q} . z - c$. sine of $\overline{p+2q} . z - d$. sine of $\overline{p+3q} . z +$ &c. $= s$ or a cos. of $pz + b$ cos. of $\overline{p+q} . z - c$ cos. of $\overline{p+2q} . z - d$. cos. of $\overline{p+3q} . z +$ &c. $= s$ the signs of the terms changing alternately two by two; then in the first case

a sine of $p'z + b$ sine of $\overline{p'+q}.z - a'$ sine of $\overline{p'+2q}.z - b'$ sine of $\overline{p'+3q}.z$ + &c. $= s'$

a sine of $p''z + b$ sine of $\overline{p''+q}.z - a''$ sine of $\overline{p''+2q}.z - b''$ sine of $\overline{p''+3q}.z$ + &c. $= s''$

&c. &c. &c. &c.

and in the second case

a cos. of $p'z + b$ cos. of $\overline{p'+q}.z - a'$ cos. of $\overline{p'+2q}.z - b'$ cos. of $\overline{p'+3q}.z$ + &c. $= s'$

a . cos. of $p''z + b$ cos. of $\overline{p''+q}.z - a''$ cos. of $\overline{p''+2q}.z - b''$ cos. of $\overline{p''+3q}.z$ + &c. $= s''$

&c. &c. &c. &c.

where the terms a, b, a', b', c', &c. a, b, a'', b'', c'', &c. &c. are formed by taking the alternate differences, as in the last theorem; p', p'', p''', &c. likewise as in that theorem, $s' = s$. 2 cos. of qz, $s'' = \overline{2 \text{ cos. of } qz}|^2$, $s''' = \overline{2 \text{ cos. of } qz}|^3$ &c.

This is plain by multiplying the series continually by 2 cos. of qz by help of lemma No. II. for case 1, and lemma No. III. for case 2.

Example. Required the sum of the series, sine of z + sine of $2z$ — sine of $3z$ — sine of $4z$ + &c.

Here $p = q = 1$ a, b, c, d, &c. $= 1, 1, 1, 1, 1$, &c. a, b, a', b', &c. $= 1, 1, 0, 0, 0$, &c. $\Big\} \therefore s' = s$. 2 cos. of z = sine of $0z$ + sine of z $\therefore s = \frac{\text{sine of } z}{2 \text{ cos. of } z}$.

Scholium 1. As the two first theorems depend on the differences of the coefficients of the immediate terms or omitting none, the two last on the differences of the coefficients of the alternate terms or omitting one term; so we may give theorems for the differences of the coefficients of the terms, omitting 2, 3, &c. terms; in fact, if r be a whole number, and the terms of the series be all positive, or any how positive and negative by sets, provided the same signs return in the same order after every set, consisting of r number of terms; by continually multiplying by 2 sine of $\frac{r}{2}qz$, we shall get new series by taking the differences of the coefficients of every term and the rth succeeding term beginning with r number of noughts; except indeed that the coefficients of the terms will sometimes have the order of signs interrupted, namely, when a greater value is to be subtracted from a less.

But if every set should have the same order to signs contrary to those in the set immediately preceding, and consequently every set omitting one set continually, have the same order of signs, then by continually multiplying by 2 cos. of $\frac{r}{2}qz$, we shall get new series by taking the differences of the coefficients of any term and the rth term from it.

Scholium II. We may by the methods above not only find the valuation of infinite series, but likewise of finite series.

Example 1, Required the sum of the r first terms of the series, cos. of $nz +$ cos. of $\overline{n+q}.z +$ cos. of $\overline{n+2q}.z$ &c.

The series *ad infinitum* may be written thus, cos. of $nz +$ cos. of $\overline{n+q}.z +$ cos. of $\overline{n+2q}.z$ - - - - $+$ cos. of $\overline{n+\overline{r-1}}.qz$ $+$ cos. of $\overline{n+rq}.z +$ cos. of $\overline{n+\overline{r+1}.q}.z +$ &c. *ad infinitum*, from which if we take cos. of $\overline{n+rq}.z +$ cos. of $\overline{n+\overline{r+1}.q}.z$ $+$ cos. of $\overline{n+\overline{r+2}.q}.z$ &c. *ad infinitum*, we shall have the required sum; the first of these by *Example* 2, *Theorem* I. $= -\frac{\text{sine of } \overline{n-\frac{1}{2}q}.z}{2 \text{ sine of } \frac{1}{2}qz}$, and the second by the same, by merely writing $n+rq$ in the room of n, is equal to $-\frac{\text{sine of }\overline{n+\overline{r-\frac{1}{2}}.q.z}}{2\text{ sine of }\frac{1}{2}qz}$, consequently the sum of the r first terms $=$

$\frac{\text{sine of }\overline{n+\overline{r-\frac{1}{2}}q.z}-\text{ sine of }\overline{n-\frac{1}{2}q}\,z}{2\text{ sine of }\frac{1}{2}qz}$.

Cor. 1. If $n=q=1$, and rz the whole circumference of the circle, we shall have cos. of $z+$ cos. of $2z+$ cos. of $3z$ - - - - $+$ cos. of $rz=\frac{\text{sine of }\overline{360^\circ+\overline{n-\frac{1}{2}}q.z}-\text{ sine of }\overline{n-\frac{1}{2}q.z}}{2\text{ sine of }\frac{1}{2}qz}=0$, a theorem said to be used by LE GENDRE in his inscription of a polygon of 17 sides; and if we have $rqz=$ to the whole circumference, we likewise have in general cos. of $nz+$ cos. of $\overline{n+q}.z$ - - - - $+$ cos. of $\overline{n+\overline{r-1}.q}.z=0$, and if $n=\frac{1}{2}q$, we have in general cos. of $nz+$ cos. of $3nz+$ cos. of $5nz+$ &c. - - - - - cos. of $\overline{2r-1}.nz=\frac{\text{sine of }2rnz}{2\text{ sine of }nz}$.

Example 2, Required the sum of the series, cos. of $nz-$ cos. of $\overline{n+q}.z+$ cos. of $\overline{n+2q}.z$ - - - - - $\pm$ cos of $\overline{n+\overline{r-1}.q}.z$ the upper sign to be taken if r be odd, and the under sign if even.

The series is evidently the difference between the series cos. of $nz-$ cos. of $\overline{n+q}.z+$ &c. *ad infinitum* and $\mp$ cos. of $\overline{n+rq}.z\pm$ cos. of $\overline{n+\overline{r+1}.q}.z$ &c. *ad infinitum*, by proper substitution in *Example* 1, *Theorem* II. we have their respective sums $\frac{\text{cos. of }\overline{n-\frac{1}{2}q}.z}{2\text{ cos. of }\frac{1}{2}qz}$ and $\mp\frac{\text{cos. of }\overline{n+\overline{r-\frac{1}{2}}.q}.z}{2\text{ cos. of }\frac{1}{2}qz}$, and the difference $=$ $\frac{\text{cos. of }\overline{n-\frac{1}{2}q}.z\pm\text{ cos. of }\overline{n+\overline{r-\frac{1}{2}}q}.z}{\text{cos. of }\frac{1}{2}qz}$, the upper sign to be taken if r be odd and the under if even.

Example 3, Required the sum of the r first terms of the series,

t^2 cos. of $pz-\overline{t+v}|^2$ cos. of $\overline{p+q}.z+\overline{t+2v}|^2$ cos. of $\overline{p+2q}.z-$ $\overline{t+3v}|^2$ cos. of $\overline{p+3q}.z$ &c.

Using *Theorem* II. to find the sum *ad infinitum*, and expanding the coefficients, we have,

$a, b, c, d,$ &c. $=t^2,\ t^2+2tv+v^2,\ t^2+4tv+4v^2,\ t^2+6tv+9v^2,\ t^2+8tv+16v^2,$ &c.
$a, a', b', c',$ &c. $=t^2,\quad 2tv+v^2,\quad 2tv+3v^2,\quad 2tv+5v^2,\quad 2tv+7v^2,$ &c.
$a, a'', b'', c'',$ &c. $=t^2, -t^2+2tv+v^2,\quad 2v^2,\quad 2v^2,\quad 2v^2,$ &c.
$a, a''', b''', c''',$ &c. $=t^2, -2t^2+2tv+v^2,\ t^2-2tv+v^2,\quad 0,\quad 0,$ &c.

Therefore s the sum *ad infinitum* $=[t^2$ cos. of $\overline{p-\frac{3}{2}q}.z+\overline{2t^2-2tv-v^2}.$ cos. of $\overline{p-\frac{1}{2}q}.z+\overline{t-v}|^2$ cos. of $\overline{p+\frac{1}{2}q}.z]\div$ $\overline{2 \text{ cos. of } \frac{1}{2}qz}|^3$, but the sum of r first terms of the series is evidently equal to the sum *ad infinitum* $\pm$ the sum of the series $\overline{t+rv}|^2$ cos. of $\overline{p+rq}.z-\overline{t+\overline{r+1}.v}|^2$ cos. of $\overline{p+\overline{r+1}.q}.z+$ &c. *ad infinitum*, which is found from the last by writing $t+rv$ for t, and $p+rq$ for p, to be $[\overline{t+rv}|^2$ cos. of $\overline{p+\overline{r-\frac{3}{2}}.q}.z+\overline{2t^2+\overline{2r-1}.2tv+\overline{2r^2-2r-1}.v^2}$ cos. of $\overline{p+\overline{r-\frac{1}{2}}.q}.z+\overline{t+\overline{r-1}.v}|^2.$ cos. of $\overline{p+\overline{r+\frac{1}{2}}.q}.z]\div\overline{2 \text{ cos. of } \frac{1}{2}qz}|^3$, which added to, or subtracted from, the value above, according as r is odd or even, gives the sum of r first terms of the original series.

Cor. If $z=0$, the cosine of any multiple of z will be equal to 1, therefore the sum of r first terms of $t^2-\overline{t+v}|^2+\overline{t+2v}|^2$ &c. will be equal to $\dfrac{t^2+2t^2-2tv-v^2+\overline{t-v}|^2}{8}\pm$

$$\frac{\overline{t+rv}|^2+2t^2+\overline{2r-1}.2tv+\overline{2r^2-2r-1}.v^2+\overline{t+\overline{r-1}.v}|^2}{8}=\frac{t^2-tv}{2}\pm$$

$\frac{t^2+\overline{2r-1}.tv+\overline{r^2-r}.v^2}{2}$ + or − to be taken according as r is odd or even, that is, $t^2+\overline{r-1}.tv+\frac{r^2-r}{2}v^2$ if r be odd, and $-rtv-\frac{\overline{r^2-r}}{2}.v^2$ if even. And thus we might proceed to the discovery of an infinite variety of theorems relative to the sines and cosines contained between any two limits in a circle, and the consequent inferences, the method being capable of a very extensive application; but rather than tire the reader's patience with what he may effect himself from what has been already said, if there should otherwise have been any difficulty, I shall propose

Theorem V.

If we have the sum of the series, a sine of $pz+b$ sine of $\overline{p+q}.z+c$ sine of $\overline{p+2q}.z+d$ sine of $\overline{p+3q}.z$ &c. expressed generally in terms of p, q, and z, we may find the sum of the series, a cos. of $pz+b$ cos. of $\overline{p+q}.z+c$ cos. of $\overline{p+2q}.z+d$ cos. of $\overline{p+3q}.z$ &c. expressed generally in terms of p, q, and z, and the contrary.

For if we put $90°+pz$ for pz in the series, and in the expression for its sum, we shall have instead of the sum of the series, a sine of $pz+b$ sine of $\overline{p+q}.z+c$ sine of $\overline{p+2q}.z$ &c., the sum of the series, a. sine of $\overline{90°+pz}+b$. sine of $\overline{90°+\overline{p+q}.z}+c$ sine of $\overline{90°+\overline{p+2q}.z}$ &c. or because the sine of an arc is equal to the sine of $\overline{180°-\text{that arc}}$, we shall have the sum of the series a. sine of $\overline{90°-pz}+b$. sine of $\overline{90°-\overline{p+q}.z}$ &c. or its equal, a. cos. of $pz+b$. cos. of $\overline{p+q}.z+c$. cos. of $\overline{p+2q}.z$ &c. which is the first part of the theorem; and by following the steps backwards, and substituting $pz - 90^\circ$ for pz throughout we get the second part Q. E. D.

This theorem evidently supposes that the *functional* values of pz are distinct in the general expression for the sum of the series, before the substitution takes place, which may not be the case if p has any particular value, or even if p, q, and z have any relation to each other.

Theorem VI.

Given the sum of the series, a sine of $\pi z + b$ sine of $\overline{\pi + \varkappa}.z + c$ sine of $\overline{\pi + 2\varkappa}.z + d$ sine of $\overline{\pi + 3\varkappa}.z +$ &c. and likewise of a cos. of $\pi z + b$ cos. of $\overline{\pi + \varkappa}.z + c$ cos. of $\overline{\pi + 2\varkappa}.z +$ &c. expressed generally and distinctly in terms of z for any particular values of π and $\varkappa$, except $\varkappa = 0$, π and $\varkappa$ having the same value in both series; there will likewise be given the sum of the series, a sine of $pz + b$ sine of $\overline{p + q}.z + c$ sine of $\overline{p + 2q}.z$ &c. and likewise of, a cos. of $pz + b$ cos. of $\overline{p + q}.z + c$ cos. of $\overline{p + 2q}.z$ &c. generally and distinctly in terms of p, q, and z.

For, calling the first series A and the second B, and putting $z = \frac{qx}{\varkappa}$, we have by substitution,

$$a \text{ sine of } \frac{q\pi}{\varkappa} x + b.\text{sine of } \overline{\frac{q\pi}{\varkappa} + q}.x + c.\text{ sine of } \overline{\frac{q\pi}{\varkappa} + 2q}.x + d.\text{ sine of } \overline{\frac{q\pi}{\varkappa} + 3q}.x + \&c. = A, \text{ and}$$

$$a \text{ cos. of } \frac{q\pi}{\varkappa} x + b.\text{cos. of } \overline{\frac{q\pi}{\varkappa} + q}.x + c.\text{cos. of } \overline{\frac{q\pi}{\varkappa} + 2q}.x + d.\text{ cos. of } \overline{\frac{q\pi}{\varkappa} + 3q}.x + \&c. = B.$$

A and B being now expressed in general terms of q and x, and particular values; multiply the first by, 2 cos. of $\overline{p - \frac{q\pi}{\varkappa}}.x$

and the second by 2 sine of $\overline{p-\frac{q\pi}{\varkappa}}\,.\,x$ by means of lemma No. II. and we get, a sine of $px-a$ sine of $\overline{p-\frac{2q\pi}{\varkappa}}.x+b$ sine of $\overline{p+q}.x-b$ sine of $\overline{p-\frac{2q\pi}{\varkappa}-q}\,.\,x+c$ sine of $\overline{p+2q}.x-c$ sine of $\overline{p-\frac{2q\pi}{\varkappa}-2q}\,.\,x$ &c.$=2$A cos. of $\overline{p-\frac{q\pi}{\varkappa}}\,.\,x$, and a sine of $px+a$ sine of $\overline{p-\frac{2q\pi}{\varkappa}}\,.\,x+b$ sine of $\overline{p+q}.x+b$ sine of $\overline{p-\frac{2q\pi}{\varkappa}-q}\,.\,x+c$ sine of $\overline{p+2q}\,.x+c$ sine of $\overline{p-\frac{2q\pi}{\varkappa}-2q}\,.x$ &c.$=2$B sine of $\overline{p-\frac{q\pi}{\varkappa}}\,x$; consequently, adding these two together, we have by dividing by 2, a sine of $px+b$ sine of $\overline{p+q}\,.x+c$ sine of $\overline{p+2q}\,.\,x$ &c.$=$A cos. of $\overline{p-\frac{q\pi}{\varkappa}}\,.\,x+$B sine of $\overline{p-\frac{q\pi}{\varkappa}}\,.\,x$, expressed generally and distinctly in terms of p, q, and x, the equation will therefore remain if we put z in the place of x throughout, and therefore the sum of the series, a sine of $pz+b$ sine of $\overline{p+q}.z$ &c. is given expressed generally in terms p, q, and of z, which is the first part of the theorem.

Again, by multiplying the series, a sine of $\frac{q\pi}{\varkappa}\,x+b$ sine of $\overline{\frac{q\pi}{\varkappa}+q}\,.\,x$ &c.$=$A, by 2 sine of $\overline{p-\frac{q\pi}{\varkappa}}\,x$, by means of lemma No. I. and the series, a cos. of $\frac{q\pi}{\varkappa}\,x+$ cos. of $\overline{\frac{q\pi}{\varkappa}+q}\,.x$ &c.$=$B, by 2 cos. of $\overline{p-\frac{q\pi}{\varkappa}}\,x$ by means of lemma No. II. we shall have a cos. of $\overline{p-\frac{q\pi}{\varkappa}}.\,x-a$ cos. of $px+b$ cos. of $\overline{p-\frac{2q\pi}{\varkappa}-q}\,.\,x-b$ cos. of $\overline{p+q}\,.\,x+$ cos. of $\overline{p-\frac{2q\pi}{\varkappa}-2q}\,.x-c$ cos. of $\overline{p+2q}\,.\,x+$ &c.$=2$A cos. of $\overline{p-\frac{q\pi}{\varkappa}}\,x$, and a cos. of $\overline{p-\frac{q\pi}{\varkappa}}\,.\,x+a$ cos. of $px+b$ cos. of $\overline{q-\frac{2q\pi}{\varkappa}-q}.x+b$ cos. of $\overline{p+q}.x+c$ cos. of $\overline{q-\frac{2q\pi}{\varkappa}-2q}$

$.x+c$ cos. of $\overline{p+2q}.x+$ &c.$=2$B cos. of $\overline{p-\frac{q\pi}{\varkappa}}x$; half the difference of these two series gives,

a cos. of $px+b$ cos. of $\overline{p+q}.x+c$ cos. of $\overline{p+2q}.x+$ &c.$=$B cos. of $\overline{p-\frac{q\pi}{\varkappa}}.x-$A sine of $\overline{p-\frac{q\pi}{\varkappa}}.x$ expressed generally and distinctly in terms of p, q, and x; and consequently by writing z for x throughout, we have the sum of the series, a cos. of $pz+b$ cos. of $\overline{p+q}.z+c$ cos. of $\overline{p+2q}.z+$ &c. expressed generally and distinctly in terms z, p, and q. Q. E. D.

Cor. I. It is evident that p and q may be taken any numbers either positive or negative, but $\varkappa$ ought not to be equal to 0, for we could not then effect the substitution $z=\frac{qx}{\varkappa}$.

Cor. II. Putting, a cos. of $pz+b$ cos. of $\overline{p+q}.z$ &c.$=$P, and a sine of $pz+b$ sine of $\overline{p+q}.z$ &c.$=$Q, and also B′ and A′ for the values that B and A become, by writing z for x in those values, that is, z for $\frac{\varkappa z}{q}$ in the given expressions B and A we shall have P$=$B′ cos. of $\overline{p-\frac{q\pi}{\varkappa}}z-$A′ sine of $\overline{p-\frac{q\pi}{\varkappa}}.z$, and Q $=$B′ sine of $\overline{p-\frac{q\pi}{\varkappa}}.z+$A′ cos. of $\overline{p-\frac{q\pi}{\varkappa}}.z$.

Cor. III. Hence we may again prove, that if we have the sum of the series, a sine of $pz+b$ sine of $\overline{p+q}.z+$ &c. expressed generally in terms of p, q, and z, we may find the series, a cos. of $pz+b$. cos. of $\overline{p+q}.z+$ &c. expressed generally in terms of p, q, and z, and the contrary. For having the sum of the first by writing π for p, $\varkappa$ for q, we shall have the sum of the series, a sine of $\pi z+b$ sine of $\overline{\pi+\varkappa}.z+$ &c. $=$A, expressed by z, and particular values; in which writing $\frac{\varkappa z}{q}$ for z, we get A′, therefore having A′ and Q, we may, by help of *Cor.* II. find P in terms of p, q, and z, and particular values, namely, the sum of the series a cos. of $pz + b$ cos. of $\overline{p+q}.z$ &c. and in a similar manner the contrary is proved. Q. E. D.

Theorem VII.

If we have the sum of the series, a sine of $pz + b$ sine of $qz + c$ sine of $rz +$ &c. expressed generally by z; we have likewise the sum of the series, a cos. of px . sine of $py + b$ cos. of qx . sine of $qy + c$ cos. of rx sine of ry &c. expressed generally by x and y. And if we have the sum of the series, a cos. of $pz + b$ cos. of $qz + c$ cos. of rz &c. expressed generally by z; we have likewise the sum of the series, a cos. of px . cos. of $py + b$ cos. of qx . cos. of $qy + c$ cos. of rx . cos. of ry &c.; and also the sum of the series, a sine of px . sine of $py + b$ sine of qx . sine of $qy + c$ sine of rx sine of ry &c. expressed generally in terms of x and y.

First; if we have the sum of the series, a sine of $pz + b$ sine of qz &c. expressed in terms of z, by writing $x+y$ in the room of z throughout, we shall have the sum of the series, a sine of $p.\overline{x+y} + b$ sine of $q.\overline{x+y} + c$ sine of $r.\overline{x+y}$ &c. expressed in terms of x and y and in like manner by writing $x-y$ for z we shall have the sum of the series, a sine of $p.\overline{x-y} + b$ sine of $q.\overline{x-y} + c$ sine of $r.\overline{x-y}$ &c. expressed in terms of x and y, therefore the half difference of these two, that is, $a.\frac{\text{sine of } p.\overline{x+y} - \text{sine of } p.\overline{x-y}}{2} + b.\frac{\text{sine of } q.\overline{x+y} - \text{sine of } q.\overline{x-y}}{2}$ $+ c.\frac{\text{sine of } r.\overline{x+y} - \text{sine of } r.\overline{x-y}}{2}$ &c. or its equal by lemma No. II, a cos. of px. sine of $py + b$ cos. of qx. sine of $qy + c$ cos. of rx.

sine of ry + &c. will likewise be expressed generally in terms of x and y, which is case the first.

Again, if we have the sum of the series a cos. of pz + b cos. of qz &c., expressed generally by z; by writing $x - y$ throughout for z we shall have the sum of the series a. cos. of $p.\overline{x-y}$ + b cos. of $q.\overline{x-y}$ + c cos. of $r.\overline{x-y}$ &c. expressed generally by x and y and by writing $\overline{x+y}$ for z throughout, we shall have the sum of of the series a cos. of $p.\overline{x+y}$ + b cos. of $q.\overline{x+y}$ + c cos. of $r.\overline{x+y}$ &c. expressed generally in terms of x and y, and consequently the half sum of the two which by lemma No. III. is equal to a cos. of px. cos. of py + b. cos. of qx. cos. of qy + c cos. of rx. cos. of ry &c. will be expressed generally in terms of x and y; and the half difference of the two which by lemma No. I. is equal to, a sine of px. sine of py + b sine of qx. sine of qy + c sine of rx. sine of ry &c. will likewise be expressed generally in terms of x and y. Q. E. D.

Corollary. From the sum of the series, a sine of pz + b sine of qz + &c. having obtained the sum of the series, a cos. of of px. sine of py + b cos. of qx. sine of qy &c. if a' be put for a cos. of px, b' for b cos. of qx, c' for c cos. of rx, &c. this series will be reduced to a' sine of py + b' sine of qy + c' sine of ry &c. which is of the first form of this theorem, and consequently from it may be deduced the sum of the series a' cos. of pw. sine of pv + b' cos. of qw. sine of qv + c cos. of rw. sine of rv &c. and therefore its equal the sum of the series a cos. of pw. cos. of px. sine of pv + b cos. of qw. cos. of qx. sine of qv + c cos. of rw. cos. of rx. sine of rv + &c. in terms of v, w, and x, but if a' had been put for a sine of py, b' for

b sine of qy &c.; we should have had the series reduced to the form a' cos. of $py + b'$ cos. of qy &c. which is of the 2d form of the theorem, and consequently from it is deduced the sum of each of the series, 1st. a' cos. of pw . cos. of $pv + b'$ cos. of qw . cos. of qv &c. that is, of the series a cos. of pw . cos. of pv . sine of $px + b$ cos. of qw . cos. of qv . sine of qx &c. in terms of v, w, and x, which is indeed similar in form to the series found by the other substitution; and 2d. the sum of the series a' sine of pw . sine of $pv + b'$ sine of qw . sine of qv &c. or its equal the sum of the series a sine of pw . sine of px . sine of $pv + b$ sine of qw . sine of qx . sine of $qv +$ &c. in terms of v, w, and x. And in a similar manner, from the sum of the series a cos. of $pz + b$ cos. of qz &c. having found the sum of the series a cos. of px . cos. of $py + b$ cos. of qx . cos. of qy &c. we may find the sum of the series a cos. of pw . cos. of pv . cos. of $px + b$ cos. of qw . cos. of qv . cos. of $qx +$ &c. in terms of w, v, and x, and likewise the sum of the series a cos. of pw . sine of pv . sine of $px + b$ cos. of qw . sine of qv . sine of qx &c. And in a similar manner also may we proceed by degrees to more complicated cases.

Example 1. Because (from *Example* 1. *Scholium* 2. after *Example* 3. *Theorem* IV.) we have the sum of the r first terms of the series, cos. of $nz +$ cos. of $\overline{n+q}.z +$ cos. of $\overline{n+2q}.z$ &c. $=$ [sine of $n + \overline{r-\frac{1}{2}.q}.z -$ sine of $\overline{n-\frac{1}{2}q}.z$] : 2 sine of $\frac{1}{2}qz$: if $x-y$ and $x+y$ be written for z, then the half sum and half difference of the resulting expressions, by case 2 of this theorem, will give the r first terms of the series cos. of nx . cos. of $ny +$ cos. of $\overline{n+q}.x$. cos. of

$\overline{n+q}\,.\,y$ &c. $= \dfrac{\text{sine of } \overline{n+\overline{r-\frac{1}{2}}\,.\,q}\,.\,\overline{x-y} - \text{sine of } \overline{n-\frac{1}{2}q}\,.\,\overline{x-y}}{4 \text{ sine of } \frac{1}{2}q\,.\,\overline{x-y}}$

$+ \dfrac{\text{sine of } \overline{n+\overline{r-\frac{1}{2}q}}\,.\,\overline{x+y} - \text{sine of } \overline{n-\frac{1}{2}q}\,.\,\overline{x+y}}{4 \text{ sine of } \frac{1}{2}q\,.\,\overline{x+y}}$; and the sum of the r first terms of the series sine of nx . sine of ny + sine of $\overline{n+q}\,.\,x$. sine of $\overline{n+q}\,.\,y$ + sine of $\overline{n+2q}\,.\,x$. sine of $\overline{n+2q}\,.\,y$ &c.

$= \dfrac{\text{sine of } \overline{n+\overline{r-\frac{1}{2}q}}\,.\,\overline{x-y} - \text{sine of } \overline{n-\frac{1}{2}q}\,.\,\overline{x-y}}{4 \text{ sine of } \frac{1}{2}q\,.\,\overline{x-y}} - [\text{sine of } \overline{n+\overline{r-\frac{1}{2}}\,.\,q}\,.\,\overline{x+y} - \text{sine of } \overline{n-\frac{1}{2}q}\,.\,\overline{x+y}] \div 4 \text{ sine of } \frac{1}{2}q\,.\,\overline{x+y}$.

Cor. If rx and ry be both multiples of the whole circumference the said two values will be equal to 0.

Example 2. Because (from *Cor.* 1. *Example* 2. *Theorem* II.) we have sine of $pz - r$ sine of $\overline{p+q}\,.\,z + r\,.\,\frac{r+1}{2}$ sine of $\overline{p+2q}\,.\,z$ &c. $= \dfrac{\text{sine of } \overline{p-\frac{1}{2}qr}\,.\,z}{2 \text{ cos. of } \frac{1}{2}qz|^{r}}$, we have by this theorem case 1, cos. of px . sine of $py - r$ cos. of $\overline{p+q}\,.\,x$. sine of $\overline{p+q}\,.\,y + r\,.\,\frac{r+1}{2}$ cos. of $\overline{p+2q}\,.\,x$. sine of $\overline{p+2q}\,.\,y$ &c. $= \dfrac{\text{sine of } \overline{p-\frac{1}{2}qr}\,.\,\overline{x+y}}{2\,.\,\overline{2 \text{ cos. of } \frac{1}{2}q\,.\,\overline{x+y}}|^{r}}$

$- \dfrac{\text{sine of } \overline{p-\frac{1}{2}qr}\,.\,\overline{x-y}}{2\,.\,\overline{2 \text{ cos. of } \frac{1}{2}q\,.\,\overline{x-y}}|^{r}}$. And because by the same cos. of $pz - r$ cos. of $\overline{p+q}\,.\,z + r\,.\,\frac{r+1}{2}$ cos. of $\overline{p+2q}\,.\,z$ &c. $= \dfrac{\text{cos. of } \overline{p-\frac{1}{2}qr}\,.\,z}{2 \text{ cos. of } \frac{1}{2}qz|^{r}}$ we have by case 2 of this theorem cos. of px . cos. of $py - r$ cos. of $\overline{p+q}\,.\,x$. cos. of $\overline{p+q}\,.\,y + r\,.\,\frac{r+1}{2}$ cos. of $\overline{p+2q}\,.\,x$. cos. of $\overline{p+2q}\,.\,y$ &c. $= \dfrac{\text{cos. of } \overline{p-\frac{1}{2}qr}\,.\,\overline{x-y}}{2\,.\,\overline{2 \text{ cos. of } \frac{1}{2}q\,.\,\overline{x-y}}|^{r}} + \dfrac{\text{cos. of } \overline{p-\frac{1}{2}qr}\,.\,\overline{x+y}}{2\,.\,\overline{2 \text{ cos. of } \frac{1}{2}q\,.\,\overline{x+y}}|^{r}}$

and sine of px . sine of $py - r$ sine of $\overline{p+q}\,.\,x$. sine of $\overline{p+q}\,.\,y$ + &c. $= \dfrac{\text{cos. of } \overline{p-\frac{1}{2}qr}\,.\,\overline{x-y}}{2\,.\,\overline{2 \text{ cos. of } \frac{1}{2}q\,.\,\overline{x-y}}|^{r}} - \dfrac{\text{cos. of } \overline{p-\frac{1}{2}qr}\,.\,\overline{x+y}}{2\,.\,\overline{2 \text{ cos. of } \frac{1}{2}q\,.\,\overline{x+y}}|^{r}}$.

Cor. If in these we either put $qr - p$ or $\frac{1}{2}qr$ in the place of p, we shall get theorems for the rectangles of sine and co-sines, rectangles of cosines and the rectangles of sines similar to those of *Cor.* II. and III. (respectively) *Example* 2. *Theorem* II. for the simple sines and cosines.

General Scholium.

It is necessary to obeserve, that there may be particular cases, in the summation of which these methods fail, and which, if not properly considered, may lead to great error, especially when new series are derived from those containing failing cases, by multiplying by fluxions, and finding the fluents of the expressions thence arising. For if the correction should happen to be sought from any of the failing cases, the summation of the new series might not only fail in the failing case of the primary expression, but in every other.

From *Example* 1. *Theorem* I. we have sine of pz + sine of $\overline{p+q}\,.\,z$ + sine of $\overline{p+2q}\,.\,z$ &c. $= \frac{\text{cos. of } \overline{p - \frac{1}{2}q}\,.\,z}{2 \text{ sine of } \frac{1}{2}qz}$; this when $z = 0$, will be sine of 0 + sine of 0 + sine of 0 &c. or 0 + 0 + 0 &c. $= \frac{\text{cos. of } 0}{2 \text{ sine of } 0} = \frac{1}{2 \text{ sine of } 0}$; that is the sum of a series of noughts infinite, which is absurd. Again, in *Example* 2. *Theorem* I. *Cor.* I. cos. of nz + cos. of $3nz$ + cos. of $5nz$ &c. $= 0$, therefore if z be taken $= 0$ it will be 1 + 1 + 1 &c. $= 0$ which ought to be infinite, and in *Cor.* II. z being $= 0$ we have $1 + 1 + 1 + 1$ &c. $= -\frac{1}{2}$.

In order to explain the reason of these absurdities, and to prevent the errors they may produce, it is necessary to consider the subject more minutely, to which purpose *Scholium* II. at the end of *Theorem* IV. will afford great assistance: from that it appears, that the sum of the r first terms of the series cos. of $nz +$ cos. of $\overline{n+q}.z +$ cos. of $\overline{n+2q}.z$ &c. $=$ $\frac{\text{sine of } \overline{n+\overline{r-\frac{1}{2}}q}.z - \text{sine of } \overline{n-\frac{1}{2}q}.z}{2 \text{ sine of } \frac{1}{2}qz}$; and by similar means that the sum of the r first terms of the series sine of $nz+$ sine of $\overline{n+q}.z+$ sine of $\overline{n+2q}z+$ &c.$=$ $\frac{-\text{cos. of } \overline{n+\overline{r-\frac{1}{2}}q}.z + \text{cos. of } \overline{n-\frac{1}{2}q}.z}{2 \text{ sine of } \frac{1}{2}qz}$: now it is plain that if qz were either equal to o or a multiple of 360°, sine of $\frac{1}{2}qz$ would be equal to o, and because r is a whole number, rqz would either be equal to o or a multiple of 360°, and consequently the sine of $\overline{n+\overline{r-\frac{1}{2}}.q}.z=$ sine of $\overline{n-\frac{1}{2}q}.z$ and the cosine of $\overline{n+\overline{r-\frac{1}{2}}.q}.z=$ cos. of $\overline{n-\frac{1}{2}.q}.z$, and therefore the sum of the series, cos. of $nz+$ cos. of $\overline{n+q}.z$ &c.$=$(when $qz=$o or some multiple of 360°) - - - - $\frac{\text{sine of } \overline{n-\frac{1}{2}q}.z - \text{sine of } \overline{n-\frac{1}{2}q}.z}{0} = \frac{0}{0}$, and of sine of $nz+$ sine of $\overline{n+q}.z$ &c.$=\frac{0}{0}$ whatever r may, whether finite or infinite. Indeed the determining the value of $\frac{0}{0}$, depends on the value of r; but if qz be any thing but o or a multiple of 360°, the value of the sine or cosine of $\overline{n+\overline{r-\frac{1}{2}}q}.z$ will depend on the value of r, and may then be varied from positive to negative and from negative to positive, by merely increasing r, and consequently when r is infinite, there being no reason for its being positive rather than negative, or negative rather than positive it should be considered o; and therefore the sum of the infinite series, cos. of $nz+$ cos. of $\overline{n+q}.z$ &c.$= -$ $\frac{\text{sine of } \overline{n-\frac{1}{2}q}.z}{\text{sine of } \frac{1}{2}qz}$ and of sine of $nz +$ sine of $\overline{n+q}.z$ &c. $=$

$\frac{\text{cos. of } \overline{n-\frac{1}{2}q}.z}{2 \text{ sine of } \frac{1}{2}qz}$ in every case, (the same as in *Example* to *Theorem* I.) except when $qz=0$ or some multiple of $360°$; on account of there being something else to be taken into consideration, in that case. Again, it appears by *Example* 2 of the said scholium, that the sum of the r first terms of the series, cos. of nz — cos. of $\overline{n+q}.z+$ cos. of $\overline{n+2q}.z$ — &c. $=$ $\frac{\text{cos. of } \overline{n-\frac{1}{2}q}.z \pm \text{ cos. of } \overline{n+\overline{r-\frac{1}{2}}.q}.z}{2 \text{ cos. of } \frac{1}{2}qz}$, the upper or under sine to be taken according as r is odd or even. And by similar means the sum of the r first terms of the series, sine of nz— sine of $\overline{n+q}.z+$ sine of $\overline{n+2q}.z$— sine of $\overline{n+3q}.z$ &c. is found $=$ $\frac{\text{sine of } \overline{n-\frac{1}{2}q}.z \pm \text{ sine of } \overline{n+\overline{r-\frac{1}{2}}.q}.z}{2 \text{ cos. of } \frac{1}{2}qz}$, the upper sign to be taken if r is odd but the under if even. Here if $qz=180°$, or any odd multiple thereof, the cosine of $\frac{1}{2}qz$ will be $=0$; and if r be even at the same time the cos. of $\overline{n+\overline{r-\frac{1}{2}}.q}.z$ will be equal to the cosine of $\overline{n-\frac{1}{2}.q}.z$ and the sine of $\overline{n+\overline{r-\frac{1}{2}}.q}.z=$ sine of $\overline{n-\frac{1}{2}qz}$; but if r be odd we shall have the cos. of $\overline{n+\overline{r-\frac{1}{2}}.q}.z$ $=$ — cos. of $\overline{n-\frac{1}{2}q}.z$ and sine of $\overline{n+\overline{r-\frac{1}{2}}.q}.z=$ — the sine of $\overline{r-\frac{1}{2}q}.z$; consequently by substituting these values in the above expressions for the sum, due regard being had to the signs, we shall find that, whatever r be, the sum of either series will be expressed by $\frac{0}{0}$: but if z be any other value it appears that $\pm$ the sine or cosine of $\overline{n+\overline{r-\frac{1}{2}}.q}.z$ depends on the value of r, and may be either positive or negative, by varying r; and consequently should as above when r is infinite be considered $=0$. And the sum of the series, sine of

$nz -$ sine of $\overline{n+q}.z +$ &c. will be $= \frac{\text{sine of } \overline{n-\frac{1}{2}q}.z}{2 \text{ cos. of } \frac{1}{2}qz}$, and of cos. of $nz -$ cos. of $\overline{n+q}.z +$ &c. $= \frac{\text{cos. of } \overline{n-\frac{1}{2}q}.z}{2 \text{ cos. of } \frac{1}{2}qz}$, as in *Example* 1, *Theorem* II. except when $qz =$ some odd multiple of 180°, something else being in that case to be taken into consideration; and thus are we to reason, in the failing cases of other expressions: but by the common rules for finding the value of an expression when the denominator and numerator vanish, we may find the value even in the failing cases; thus by dividing the fluxion of the numerator by the fluxion of the denominator in the expressions $\frac{-\text{cos. of } \overline{n+r-\frac{1}{2}q}.z + \text{cos. of } \overline{n-\frac{1}{2}q}.z}{2 \text{ sine of } \frac{1}{2}qz}$ and $\frac{\text{sine of } \overline{n+r-\frac{1}{2}.q}.z - \text{sine of } \overline{n-\frac{1}{2}q}.z}{2 \text{ sine of } \frac{1}{2}qz}$, and then making $qz = 0$, or some multiple of 360°, we shall get simply, 0 for the sum of the r first terms of the series sine of $nz +$ sine of $\overline{n+q}.z +$&c., and r for the sum of the r first terms of the series cos. of $nz +$ cos. of $\overline{n+q}.z +$ &c. when $z = 0$ or some multiple of 360°, that is, 0 for the sum of the r first terms of the series, $0+0+0$ &c. and r for the sum of the r first terms of the series, $1+1+1$ &c. which is self-evident. And thus may we proceed in other expressions when the sum of r terms can be obtained by a general value.

That these things should happen as above described, is likewise evident, from the investigations of the theorems; for in *Theorem* I. for instance, we have $s^{\pi} = \pm \overline{2 \text{ sine of } \frac{1}{2}qz}|^{\pi-1}$ or $\pm s . \overline{2 \text{ sine of } \frac{1}{2}qz}|^{\pi}$, π being a positive whole number; therefore if the sine of $\frac{1}{2}qz$ be $= 0$, which will happen when $qz = 0$ or some multiple of 360°, it is plain that we should have

$s^{\varpi} = \pm s' \times 0$, or $\pm s \times 0$, and consequently s^{ϖ} ought likewise to come out equal to 0, and therefore s, would be $= \frac{0}{0}$; and consequently when s^{ϖ} in that case does not come out $= 0$, it is certain that there must have been something neglected: and to shew how this may happen, we observe that since *Theorem* I. and II. require the differences of the coefficients of every term and the next succeeding term to be taken, it is evident that the last term will have nothing to be taken from, and will consequently remain through every new series; in consequence of which there will be terms of the form Δ . sine or cosine of $\overline{\varpi + qr}.z$, (in which r is a whole number and infinite, the number of terms of the series being infinite,) whose coefficient Δ will never be $= 0$ unless the series a, b, c, &c. be converging: these terms are unavoidably omitted, by reason of their place being at an infinite distance, and can consequently never be arrived at; but still unless it be equal to 0, it should not be omitted; which it cannot be unless, either in the above mentioned circumstance of the series a, b, c, &c. being converging, or when the terms of the series of sines or cosines, are continually changing their signs, for different values of r; which it will always do when qz is not equal to 0 or some multiple of 360°; provided the coefficients a, b, c, &c. are all affirmative: and consequently the said terms may be omitted in every such case, there being no reason for taking one sign rather than the other: but if qz were equal to 0 or some multiple of 360°, since Δ . sine or cosine of $\overline{\varpi + qr}.z$ will then be simply Δ . sine or cos. of pz, and therefore if the same sign whatever r may be, when a, b, c, d, &c. ----- to Δ, have all the same signs; and consequently cannot be then neglected unless in the case above mentioned of a, b, c, &c. being converging, in which circumstance it will have no failing case: but had the coefficients a, b, c, d, &c. been alternately $+$ and $-$, the failing case would not happen when $qz = 0$ or a multiple of 360°, for then there being no reason for taking Δ of one sign rather than the other, it should therefore be taking equal to 0; but it will happen when sine or cos. of $\overline{\varpi + qr.z}$ is alternately positive and negative, by continually increasing r by 1: for then the coefficients of the terms of the form, sine or cos. of $\overline{\varpi + qr}.z$ being alternately positive and negative; and likewise the terms themselves alternately positive and negative, the whole values resulting from them will have the same determinate sign, and this will be when $qz = 180°$ or some multiple thereof. And if a, b, c, d, &c. be positive and negative according to some other law, the failing cases may be found by the like reasoning; which is likewise applicable to the other theorems.

These remarks pave the way to the correction of fluents necessary in the application of the doctrine of fluxions to these series.

1. In *Example* 2, *Theorem* I. if for nz we write $k+z$, and for, q we write 1, we shall have cos. of $\overline{k+z}+$ cos. of $\overline{k+2z}$ $+$ cos. of $\overline{k+3z}$ &c. $= -\frac{\text{sine of } \overline{k+\frac{1}{2}z}}{2 \text{ sine of } \frac{1}{2}z} = -$ sine of $k . \frac{\text{cos. of } \frac{1}{2}z}{2 \text{ sine of } \frac{1}{2}z}$ $-\frac{\text{cos. of } k . \text{sine of } \frac{1}{2}z}{2 \text{ sine of } \frac{1}{2}z} = -\frac{\text{sine of } k . \text{ cos. of } \frac{1}{2}z}{2 \text{ sine of } \frac{1}{2}z} - \frac{\text{cos. of } k}{2}$. Multiply both sides by $\dot{z}$, and find the consequent fluents, and we shall have sine of $\overline{k+z} + \frac{\text{sine of } \overline{k+2z}}{2} + \frac{\text{sine of } \overline{k+3z}}{3}$ &c. $=$ fluent of $[-\frac{\dot{z}}{2}$ sine of $k . \frac{\text{cos. of } \frac{1}{2}z}{\text{sine of } \frac{1}{2}z} - \frac{\text{cos. of } k}{2} \dot{z}]$, which because $\frac{\dot{z}}{2}$ cos. of

$\frac{1}{2}z$, is equal to the fluxion of sine of $\frac{1}{2}z$, is equal to the fluent of $[-\text{sine of } k . \frac{\text{fluxion of sine of } \frac{1}{2}z}{\text{sine of } \frac{1}{2}z} - \frac{\text{cos of } k}{2} . \dot{z}] = -\text{sine of } k .$ log. sine of $\frac{1}{2}z - \frac{\text{cos. of } k}{2} . z +$ a correction: now this correction must not be sought when $z = 0$ or a multiple of $360°$: for in that case from what has been just now said, the primary equation fails, or rather there is a supplemental value only then to be prefixed; therefore the easiest method which offers, is when $z = 180°$, we then have the sine of $\overline{k+z} = -k$, sine of $\overline{k+2z} =$ sine of k, sine of $\overline{k+3z} = -$ sine of k &c. and sine of $\frac{1}{2}z = 1$, consequently putting Q for $\frac{1}{4}$ of the periphery of a circle whose radius one, the expression will become $-$ sine $k + \frac{\text{sine of } k}{2} - \frac{\text{sine of } k}{3} + \frac{\text{sine of } k}{4}$ &c. or sine of $k \times$ log. of $\frac{1}{2} =$ $-$ cos. of $\overline{k|} . Q +$ correction $\therefore$ correction $=$ sine of k . log. of $2 +$ cos. of $\overline{k|} . Q$, which correction being prefixed we have, sine of $\overline{k+z} + \frac{\text{sine of } \overline{k+2z}}{2} + \frac{\text{sine of } \overline{k+3z}}{3}$ &c.$=$ sine of $\overline{k} \times$ log. of $\overline{\frac{1}{2 \text{ sine of } \frac{1}{2}z}} + Q - \frac{z}{2} \times$ cos. of k: which is only true whilst z is between 0 and $360°$; for though our primary equation fails only when z is 0 or some multiple of $360°$, and is true in every other case, whatever z may be, whether more or less than $360°$; still it cannot be so in this derivative equation: for suppose K to be the said supplemental value, which is equal to 0 in every other case but that mentioned above, the derivative expression, will in that case contain the supplement, the fluent of $K . \dot{z}$ producing a correction which will remain when it is once introduced, though K may afterwards vanish, namely, when z becomes neither 0, nor any multiple of $360°$

and therefore every time z becomes, by flowing, any multiple of 360°; K being introduced, it will introduce an additional correction, which will remain afterwards.

If this equation be multiplied by $\dot{z}$ and the fluent be again taken, we shall have $\frac{\text{cos. of } \overline{k+z}}{1} + \frac{\text{cos. of } \overline{k+2z}}{4} + \frac{\text{cos. of } \overline{k+3z}}{9}$ &c. $= -$ fluent of [sine of $\bar{k} \cdot \dot{z}$ log. of $\overline{\frac{1}{2 \text{ sine of } \frac{1}{2}z} + Q - \frac{z}{2}} \cdot \dot{z}$ cos. of k] which fluent is easily found by infinite series, but if k be $= 0$ we shall have cos. of $k = 1$, and the fluent $= Qz - \frac{z^2}{4}$ independent of the correction, that is cos. of $z + \frac{\text{cos. of } 2z}{2^2} + \frac{\text{cos. of } 3z}{3^2}$ &c. $= A - Qz + \frac{z^2}{4}$, A standing for the correction: if z be $=$ to the arc of 90° or Q, we shall have cos. of $z = 0$, cos. of $2z = -1$, cos. of $3z = 0$, cos. of $4z = 1$ &c. therefore we shall have by substitution $-\frac{1}{2^2} + \frac{1}{4^2} - \frac{1}{6^2} + \frac{1}{8^2}$ &c. $= A - Q^2 + \frac{Q^2}{4} = A - \frac{3}{4}Q^2$, but if in the equation z be taken $= 180°$ or $2Q$, we shall have cos. of $z = -1$, cos. of $2z = +1$, cos. of $3z = -1$, &c. &c. $\therefore -1 + \frac{1}{2^2} - \frac{1}{3^2} +$ &c. $= A - 2Q^2 + Q^2$ or $A - Q^2$, which series being the same as the other series when multiplied by 4, we have $A - Q^2 = 4A - 3Q^2 \therefore A = \frac{2}{3}Q^2 \therefore 1 - \frac{1}{2^2} + \frac{1}{3^2} -$ &c. $= Q^2 - A = \frac{Q^2}{3}$, and cos. of $z + \frac{\text{cos. of } 2z}{4} + \frac{\text{cos. of } 3z}{9}$ &c. $= \frac{2}{3}Q^2 - Qz + \frac{z^2}{4}$. It is remarkable that this equation is true, not only when the equation from which it is derived is true, but likewise when $z = 0$ or 360° in which that fails, and that the correction might have been sought in those cases had this circumstance been known. Multiply this again by $\dot{z}$, and find the fluent, and we have sine of $z + \frac{\text{sine of } 2z}{8} + \frac{\text{sine of } 3z}{3^3}$ &c. $= \frac{2}{3} Q^2 z - \frac{Qz^2}{2} + \frac{z^3}{12}$ and this requires no correction whilst z is from 0 to 360 inclusively of both; this is evident at first sight, as we are not now obliged as before to avoid correcting when $z=0$ or $360°$, as the equation from which this is derived does not fail in those cases.

2. If in the equation sine of nz — sine of $\overline{n+q}.z. +$ sine of $\overline{n+3qz}$ — &c. $= \frac{\text{sine of } \overline{n-\frac{1}{2}q}z}{2 \text{ cos. of } \frac{1}{2}qz}$, failing when $qz = 180°$ or an odd multiple thereof, we put $n=1$, $q=2$ we have sine of z — sine of $3z+$ sine of $5z$ &c. $= 0$, failing when $z = 90°$ or any odd multiple thereof; if we multiply this by $\dot{z}$ and find the fluent we shall have cos. of $z - \frac{\text{cos. of } 3z}{3} + \frac{\text{cos. of } 5z}{5}$ — &c. = correction, which must not be sought when $z = 90°$, or odd multiple thereof; if it be sought when $z = 0$, we shall have it $= 1 - \frac{1}{3} + \frac{1}{5}$ &c. $= \frac{1}{2} Q$, which will answer whilst z is exclusively from $-90°$ to $+90°$. If the correction had been sought when $z=180°$ we should have it $= -1 + \frac{1}{3} - \frac{1}{5}$ &c. $= -\frac{1}{2} Q$, answering whilst z is from 90° to 270°.

3. Again, from *Cor.* 1. *Example* 2. *Theorem* I. cos. of $z+$ cos. of $3z+$ cos. of $5z+$ &c. is equal to 0, failing (from above) when $qz = 0$ or a multiple of 360°, and therefore when $z = 0$ or a multiple of 180°: if we multiply by the $\dot{z}$ and take the fluent we have sine of $z + \frac{\text{sine of } 3z}{3} + \frac{\text{sine of } 5z}{5}$ &c. = correction, which should not be sought when $z = 0$ or any multiple of 180, when $z = 90°$ it becomes sine of $90° + \frac{\text{sine of } 3 \times 90°}{3} + \frac{\text{sine of } 5 \times 90°}{5}$ &c. that is $1 - \frac{1}{3} + \frac{1}{5}$ &c. or its equal $\frac{Q}{2}$ for the

correction, the same as LANDEN finds, this is true whilst z is exclusively between o and 180°.

4. In *Cor.* II. *Example* 1. *Theorem* I. p being $= 1$ we have sine of z + sine of $3z$ + sine of $5z$ &c. $= \frac{1}{2 \text{ sine of } z}$ failing (from above) when qz or $2z = 0$, or a multiple of 360°, and therefore when $z = 0$, or a multiple of 180°; if we multiply this by $\dot{z}$ and find the fluent we have, because $\frac{\dot{z}}{2 \text{ sine of } z}$ (by putting y for the sine of z) $= \frac{y^{-1}\dot{y}}{2\sqrt{1-y^2}} = \frac{y^{-2}\dot{y}}{2\sqrt{y^{-2}-1}} = -\frac{\dot{x}}{2\sqrt{x^2-1}}$ (x being put for $\frac{1}{y}$) whose fluent is $= -\frac{1}{2}$ log. of $x + \sqrt{x^2-1}$ $= -\frac{1}{2}$ log. of $\frac{1+\sqrt{1-y^2}}{y}$, consequently cos. of z + $\frac{\text{cos. of } 3z}{3}$ + $\frac{\text{cos. of } 5z}{5} = \frac{1}{2}$ log. of $\frac{1+\sqrt{1-y^2}}{y}$ + correction, which being sought when $z = 90°$ and consequently $y = 1$ and the cosines of z, of $3z$, of $5z$, &c. $= 0$, we have it equal to o; and this has no failing case since it will not fail when $z = 0$ or any multiple of 180° in which primitive equation does. If z be $= 45°$ we shall have cos. of $z = \sqrt{\frac{1}{2}}$, cos. of $3z = -\sqrt{\frac{1}{2}}$, cos. of $5z = -\sqrt{\frac{1}{2}}$, cos. of $7z = +\sqrt{\frac{1}{2}}$, cos. of $9z = +\sqrt{\frac{1}{2}}$, &c. therefore $\sqrt{\frac{1}{2}} \times : 1 - \frac{1}{3} - \frac{1}{5} + \frac{1}{7} + \frac{1}{9} -$ &c. or $\sqrt{\frac{1}{2}} \times 1 - \frac{8}{3.5} + \frac{16}{7.9} - \frac{24}{11.13}$ &c. $= \frac{1}{2}$ log. of $\frac{1+\sqrt{\frac{1}{2}}}{\sqrt{\frac{1}{2}}}$ $\therefore$ $1 - \frac{8}{3.5} + \frac{16}{7.9}$ &c. $= \sqrt{\frac{1}{2}}$. log. of $\frac{1+\sqrt{\frac{1}{2}}}{\sqrt{\frac{1}{2}}} = \sqrt{\frac{1}{2}}$ log. of $\sqrt{2} + 1$ $\therefore$ $\frac{1}{3.5} - \frac{2}{7.9} + \frac{3}{11.13}$ &c. $= \frac{1}{8} - \frac{\sqrt{\frac{1}{2}}}{8}$ log. of $\sqrt{2} + 1$.

5. By *Theorem* I. *Example* 6, we have the sum of the series sine of $pz + g$ sine of $\overline{p+q}.z + g^2$ sine of $\overline{p+2q}.z$ + &c.

$= \frac{-g \text{ sine of } \overline{p-q}\,.\,z + \text{sine of } pz}{g^2+1-2g \text{ cos. of } qz}$, if this should have a failing case, it will be, by this scholium, when $qz = 0$ or some multiple of 360° provided g be affirmative; but if g be negative, it will be when qz is some odd multiple of 180°; a similar expression to this is given by Mr. LANDEN by his method.

If p be $= q = 1$ we shall have, sine of $z + g$ sine of $2z + g^2$ sine of $3z$ &c. $= \frac{\text{sine of } z}{g^2+1-2g \text{ cos. of } z}$, if we now multiply by $\dot{z}$ calling the cosine of z, x and find the fluent, we shall have cos. of $z + \frac{g \text{ cos. of } 2z}{2} + \frac{g^2 \text{ cos. of } 3z}{3}$ &c. $= \frac{1}{2g}$. log. of $\overline{1+g^2-2gx}$ which has no failing case.

6. According to this *General Scholium, Example* 2. to *Theorem* II. has failing cases in the investigation, unless the series $1, r, r.\frac{r+1}{2}$ &c. converge; thus those in the *Corollaries* I. II. and III. when qz is any odd multiple of 180°. By bringing both series to one side in the equations in *Cor.* II. we have sine of $\overline{qr-p}\,.\,z$ + sine of $pz - r \times$ sine of $q.\overline{r+1}-p\,.\,z$ + sine of $\overline{p+q}\,.\,z$ + &c. $= 0$, and cos. of $\overline{qr-p}\,.\,z$ − cos. of $pz - r \times$ cos. of $\overline{q.\overline{r+1}-p}\,.\,z$ − cos. of $\overline{p+q}\,.\,z$ + &c. $= 0$. Multiply them both by $\dot{z}$ and take the correct fluents when $z = 0$, and we get

$$\frac{\text{cos. of } \overline{qr-p}\,.\,z}{qr-p} + \frac{\text{cos. of } pz}{p} - r.\frac{\text{cos. of } q.\overline{r+1}-p\,.\,z}{q.\overline{r+1}-pz} + \frac{\text{cos. of } \overline{p+q}\,.\,z}{p+q} + r.\frac{r+1}{2}.\frac{\text{cos. of } q.\overline{r+2}-p\,.\,z}{q.\overline{r+2}-pz} + \frac{\text{cos. of } \overline{p+2q}\,.\,z}{p+2q} \text{ \&c.} = M,$$

and

$$\frac{\text{sine of } \overline{qr-p}\,.\,z}{qr-p} - \frac{\text{sine of } pz}{p} - r.\frac{\text{sine of } q.\overline{r+1}-p\,.\,z}{q.\overline{r+1}-p} - \frac{\text{sine of } \overline{p+q}\,.\,z}{p+q} + \text{\&c.} = 0,$$

M being put for $\frac{1}{qr-p} + \frac{1}{p} - r.\frac{1}{q.\overline{r+1}-p} + \frac{1}{p+q} + r.\frac{r+1}{2}.\frac{1}{q.\overline{r+2}-p} + \frac{1}{p+2q}$ &c. $= \frac{qr}{p.\overline{qr-p}} - r.\frac{\overline{r+2}.q}{\overline{p+q}.\overline{q.\overline{r+1}-p}}$

$+ r . \frac{r+1}{2} . \frac{\overline{r+4 . q}}{\overline{p+2q . q . r+2-p}}$ &c., if we now multiply the first of these by sine of $\overline{\frac{1}{2}qr-p} . z$, and the second by cos. of $\overline{\frac{1}{2}qr-p} . z$, and take the difference we have

$\frac{\text{sine of } \overline{\frac{1}{2}qr-p} . z \times \text{cos. of } \overline{qr-pz} - \text{cos. of } \overline{\frac{1}{2}qr-p} . z \times \text{sine of } \overline{qr-pz}}{qr-p} + [\text{sine of } \overline{\frac{1}{2}qr-p} . z . \text{cos. of } pz + \text{cos. of } \overline{\frac{1}{2}qr-pz} . \text{sine of } pz] \div p - \frac{r}{\overline{q . r+1-p}} \times [\text{sine of } \overline{\frac{1}{2}qr-pz} . \text{cos. of } \overline{r+1 . q-p} . z - \text{cos. of } \overline{\frac{1}{2}qr-p} . z . \text{sine of } \overline{r+1 . q-p . z}] - \frac{r}{p+q} \times [\text{sine of } \overline{\frac{1}{2}qr-p} . z . \text{cos. of } \overline{p+q} . z + \text{cos. of } \overline{\frac{1}{2}qr-p} . z \times \text{sine of } \overline{p+q} . z]$ &c.

which by trigonometry is reducible to $- \frac{\text{sine of } \frac{1}{2}qr . z}{qr-p} + \frac{\text{sine of } \frac{1}{2}qrz}{p} + r \frac{\text{sine of } q . \overline{\frac{1}{2}r+1} . z}{\overline{q . r+1-p}} - r \frac{\text{sine of } q . \overline{\frac{1}{2}r+1} . z}{p+q}$ &c. $=$ M . sine of $\overline{\frac{1}{2}qr-pz}$; or $\frac{\text{sine of } \frac{1}{2}qrz}{p . \overline{qr-p}} - r \frac{\text{sine of } q . \overline{\frac{1}{2}r+1} . z}{\overline{p+q} . \overline{q . r+1-p}} = \text{M} . \frac{\text{sine of } \overline{\frac{1}{2}qr-p} . z}{qr-2p}$, the same as LANDEN finds, page 83, Mem. 5. We may farther add, that when series are obtained from others having failing cases by substitution, as in *Scholium* III. to *Theorem* I. or as in *Theorem* V. and VI. regard should be had to those failing cases, according to the manner of substitution, in order to find the failing cases in the new series. We might proceed to many more examples, in finding the sums of new series from others multiplied by fluxions, or we might give examples of finding the sums of new series by throwing others into fluxions: but my chief object in these latter examples was to obviate any difficulty that might appear in choosing the cases for the correction of the fluents. There are other inferences to be drawn, which I may perhaps consider at some future period.